Supplementary Information for
An anomalous topological phase transition in spatial random graphs

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Contents

Supplementary Note 1 Analytics

In the following section we will first present some preliminaries about the $\mathbb{S}^1$-model. We will do so from the point of view of network theory, focussing on connection probabilities and hidden variables of nodes. This section functions as a summary of known results about the $\mathbb{S}^1$-model. For further reading about this model and its alternative formulation, the $\mathbb{H}^2$-model, we kindly direct the reader to [1]. We will then look at the model from another direction, focussing on the fact that the network can be represented as a gas of fermions (links), where the nodes of the network define the available states. This will allow us to fully determine the thermodynamics of the model, generalizing what was already done in the main text of the paper. We investigate the surprising result further by showing that many can be recovered by a highly simplified toy-model. We then revert back to the network point of view to find the finite size scaling behaviour of the clustering coefficient in different cases. This is done by looking at $N \gg 1$ but finite, taking only the dominant contributions into account. We distinguish between the cases $\beta < \beta_c$ and $\beta = \beta_c$ as these show different behaviour. Finally we will study how the clustering coefficient in the thermodynamic limit ($N \to \infty$) approaches zero in the limit $\beta \to \beta_c^+$. Notation-wise we choose to use $a \simeq b$ to refer to 'a is asymptotically equivalent to b' and $\sim$ to 'a is asymptotically proportional to b'.

Supplementary Note 1.1 Preliminaries

The average clustering coefficient for a node with hidden degree κ and angular position θ is defined in the $\mathbb{S}^1$ [2] model as

$$\overline{c}(\kappa,\theta) = \left(\frac{N}{\overline{k}(\kappa,\theta)}\right)^2 \iiiint \mathrm{d}\kappa'\mathrm{d}\kappa''\mathrm{d}\theta'\mathrm{d}\theta''\rho(\kappa')\rho(\kappa'')p(\kappa',\kappa'',\theta',\theta'')p(\kappa,\kappa'',\theta,\theta'')p(\kappa,\kappa',\theta,\theta'), \tag{S1}$$

in a network with system size N. Here the function $\overline{k}(\kappa,\theta)$ is the average degree of a node with hidden coordinates (κ,θ), $\rho(\kappa)$ is the hidden degree density and $p(\kappa,\kappa',\theta,\theta'')$ is the connection probability between two nodes with hidden coordinates (κ,θ) and (κ',θ'). The exact form of these functions will be discussed in the following. Note that as the model has rotation symmetry, one only needs to investigate the node at angular coordinate $\theta = 0$. The average clustering coefficient can be computed from $\overline{c}(\kappa)$ in the following manner

$$\overline{c} = \int \mathrm{d}\kappa'\rho(\kappa')\overline{c}(\kappa'). \tag{S2}$$

However, as $\overline{c}(\kappa)$ is a bounded monotonically decreasing function, it suffices to find the scaling of $\overline{c}(\kappa)$ for small κ [3]. In Eq. (S1), $\rho(\kappa)$ defines the distribution of the hidden degrees. In the following, we always apply a power law distribution. As

we are interested in finite-sized scale-free networks, we choose the following distribution

$$\rho(\kappa) = \begin{cases} \frac{(\gamma-1)\kappa_0^{\gamma-1}}{1-\left(\frac{\kappa_c}{\kappa_0}\right)^{1-\gamma}}\kappa^{-\gamma} & \text{if} \quad \kappa_0 \leq \kappa \leq \kappa_c \\ 0 & \text{else} \end{cases} \tag{S3}$$

Note that from this expression also the homogeneous distribution can be reached. This can be done in two different ways. The first is by letting $\kappa_c \to \kappa_0$. Note that then $\rho(\kappa) \to \infty$ if $\kappa_0 \leq \kappa \leq \kappa_c$, but that this region also goes to zero width. Thus, we end up with a delta distribution (note that the integral of $\rho(\kappa)$ always gives 1, irrespective of κ_c), exactly what we want for a homogeneous distribution. We can then set $\kappa_0 = \langle k \rangle$ to obtain the correct average degree. One can make similar arguments by leaving $\kappa_c > \kappa_0$ and $\gamma \to \infty$. In this case, one again ends up with the same distribution. We choose to not specify the specific form of the cut-offs just yet. We just demand that κ_0 is such that the correct average degree is obtained and that, to lowest order, it does not depend on the system size. The average degree of nodes with hidden variable κ and angular position θ is defined as

$$\bar{k}(\kappa, \theta) = N \iint \mathrm{d}\kappa' \mathrm{d}\theta' \rho(\kappa') p(\kappa, \kappa', \theta, \theta'). \tag{S4}$$

The function p describes the probability of two nodes in the network being connected and is given by the Fermi-Dirac distribution. In Ref. [4] it is noted that for $\beta > \beta_c$, one can define the connection probability in the thermodynamic limit, given in terms of the spatial coordinates r, in our 1D case the coordinates on a infinite line:

$$p(\kappa', \kappa'', r', r'') = \frac{1}{1 + \left(\frac{|r' - r''|}{\hat{\mu}\kappa'\kappa''}\right)^{\beta}}, \tag{S5}$$

Here $\hat{\mu} = \exp \mu$ where μ is the chemical potential which fixes the average degree of the network. We come back to this shortly. As was noted in the main text, the relation between the coordinate θ on a circle with a finite radius and the coordinate on the real line r is $r = \frac{N\theta}{2\pi}$. So for finite sizes this becomes

$$p(\kappa', \kappa'', \theta', \theta'') = \frac{1}{1 + \left(\frac{N\Delta\theta}{2\pi\hat{\mu}\kappa'\kappa''}\right)^{\beta}}. \tag{S6}$$

Here $\Delta\theta = \pi - |\pi - |\theta' - \theta''||$. To find the value of $\hat{\mu}$ we demand that the average degree remains constant:

$$\langle k \rangle = \frac{N}{\pi} \iint \mathrm{d}\kappa' \mathrm{d}\kappa'' \rho(\kappa')\rho(\kappa'') \int_0^{\pi} \mathrm{d}\theta' \frac{1}{1 + \left(\frac{N\theta'}{2\pi\hat{\mu}\kappa'\kappa''}\right)^{\beta}} \tag{S7}$$

$$= N \iint \mathrm{d}\kappa' \mathrm{d}\kappa'' \rho(\kappa')\rho(\kappa'') \, {}_2F_1 \left[\begin{matrix} 1, 1/\beta \\ 1 + 1/\beta \end{matrix} ; -\left(\frac{N}{2\hat{\mu}\kappa'\kappa''}\right)^{\beta} \right] \tag{S8}$$

$$\simeq \frac{2\pi\hat{\mu}\langle k\rangle^2}{\beta\sin(\pi/\beta)} + (2\hat{\mu})^\beta \frac{N^{1-\beta}}{1-\beta}\kappa_0^{2\beta}\left(\frac{\gamma-1}{\gamma-\beta-1}\right)^2 \tag{S9}$$

Here, ${}_2F_1\left[\begin{array}{c} a,b \\ c \end{array};z\right]$ is the ordinary hypergeometric function [1]. Which one of these terms is dominant depends on β. If $\beta > 1$, the first term is more important and so we can isolate $\hat{\mu}$ to obtain

$$\hat{\mu} \simeq \frac{\beta\sin(\pi/\beta)}{2\pi\langle k\rangle}. \tag{S10}$$

If $\beta < 1$, the second term dominates and we obtain

$$\hat{\mu} \simeq \frac{1}{2\kappa_0^2}(1-\beta)^{1/\beta}\langle k\rangle^{1/\beta}N^{1-1/\beta}\left(\frac{\gamma-\beta-1}{\gamma-1}\right)^{2/\beta}. \tag{S11}$$

However, as explained in the main text, using connection probability (S6) also when $\beta < 1$ leads to an ever more homogeneous network. If instead we want to preserve the degree sequence also below the critical β we need to redefine the connection probability in this regime:

$$p(\kappa',\kappa'',\theta',\theta'') = \frac{1}{1+\dfrac{(N\Delta\theta)^\beta}{(2\pi)^\beta\hat{\mu}\kappa'\kappa''}}. \tag{S12}$$

If we use this connection probability instead in Eq. (S7) we obtain for $\hat{\mu}$ when $\beta < 1$

$$\hat{\mu} \simeq \frac{(1-\beta)}{2^\beta\langle k\rangle N^{1-\beta}}. \tag{S13}$$

It can be shown that higher order terms become relevant when $\beta \to 1$. Thus, when $\beta = 1$ the form of the dominant contribution to chemical potential will change. Note that in this case both connection probabilities are equivalent. To fix the average degree we write

$$\begin{aligned}\langle k\rangle &= \frac{N}{\pi}\iint \mathrm{d}\kappa'\mathrm{d}\kappa''\rho(\kappa')\rho(\kappa'')\int_0^\pi \mathrm{d}\theta'\frac{1}{1+\left(\frac{N\theta'}{2\pi\hat{\mu}\kappa'\kappa''}\right)} \\ &= N\iint \mathrm{d}\kappa'\mathrm{d}\kappa''\rho(\kappa')\rho(\kappa'')\,{}_2F_1\left[\begin{array}{c} 1,1 \\ 2 \end{array};-\frac{N}{2\hat{\mu}\kappa'\kappa''}\right] \simeq 2\langle k\rangle^2\hat{\mu}\ln(N)\end{aligned} \tag{S14}$$

This then leads to

$$\hat{\mu} \simeq \Big(2\langle k\rangle\ln(N)\Big)^{-1}. \tag{S15}$$

Having derived the expressions above, we can also determine $\overline{k}(\kappa,\theta)$. For large N, Eq. (S4) evaluates to $\overline{k}(\theta,\kappa) \simeq \mathcal{C}\hat{\mu}\langle\kappa\rangle\kappa$, where $\mathcal{C}$ is some constant that depends on β. Thus, we note that the expected degree is proportional to the hidden degree.

[1] Note that we use a slightly different form than the standard ${}_2F_1[a,b;c;z]$ for aesthetic purposes.

Now, integrating over κ we obtain $\langle k\rangle \simeq \mathcal{C}\hat{\mu}\langle\kappa\rangle^2$, where above we have defined the various $\hat{\mu}$'s s.t. $\langle k\rangle = \langle\kappa\rangle$. This then implies that $\overline{k}(\theta,\kappa) \simeq \kappa$, i.e. that the hidden degree exactly represents the expected degree of a node.

Supplementary Note 1.2 The network as a gas of fermions

We will now look at the network in a different picture, using the fact that, as explained in the main text, the edges can be seen as fermions, with occupation numbers given by $(1+\exp(\beta(\epsilon-\mu)))^{-1}$.

Supplementary Note 1.2.1 The density of states

We start from the most general form of the connection probability

$$p = \frac{1}{1+e^{\beta(\epsilon-\mu)}}.$$

Now, if we want the connection probability to have the form as given in Eq. (S6), where $\hat{\mu} = \exp(\mu)$, we must define the energy per link/particle as

$$\epsilon(\theta',\theta'',\kappa',\kappa'') = \ln\left(\frac{N(\pi-|\pi-|\theta'-\theta''||)}{2\pi\kappa'\kappa''}\right). \tag{S16}$$

As was mentioned above as well as in the main text, we must change the form of the connection probability for $\beta < \beta_c$ in order to have a degree distribution independent of temperature. The form we then use is that given in Eq. (S12), where $\hat{\mu} = \exp\beta\mu$, which leads to the following energy per particle.

$$\epsilon(\theta',\theta'',\kappa',\kappa'') = \ln\left(\frac{N(\pi-|\pi-|\theta'-\theta''||)}{2\pi(\kappa'\kappa'')^{1/\beta}}\right). \tag{S17}$$

Note that, from a standard statistical physics perspective having the energy levels depend on temperature explicitly is unusual. In fact, we will see that we need to be very careful when deriving the thermodynamic properties. However, we will also show that, from a network perspective, the results we obtain are completely valid.

With the two expressions for the energy per particles we can then derive the density of states as follows:

$$\rho(\epsilon) = \int_0^{2\pi} \mathrm{d}\theta'\rho(\theta')\int_0^{2\pi}\mathrm{d}\theta''\rho(\theta'')\int_{\kappa_0}^{\infty}\mathrm{d}\kappa'\rho(\kappa')\int_{\kappa_0}^{\infty}\mathrm{d}\kappa''\rho(\kappa'')\delta(\epsilon-\epsilon(\theta',\theta'',\kappa',\kappa'')) \tag{S18}$$

This leads to two distinct density of states. The first, using Eq. (S16), is

$$\rho(\epsilon) = 2\left(\frac{\gamma-1}{2-\gamma}\right)^2\kappa_0^4 e^{\epsilon+\epsilon_{\text{max}}}\Theta(\epsilon_{\text{max}}-\epsilon)\left[1+e^{(2-\gamma)(\epsilon_{\text{max}}-\epsilon)}((2-\gamma)(\epsilon_{\text{max}}-\epsilon)-1)\right], \tag{S19}$$

with $\epsilon_{\text{max}} = \ln\left(\frac{N}{2\kappa_0^2}\right)$ (connecting two points on opposite sides of the $\mathbb{S}^1$ manifold with both points having the minimal expected degree) and the second, using Eq. (S17), is

$$\rho(\epsilon) = 2\left(\frac{\gamma-1}{1+1/\beta-\gamma}\right)^2 \kappa_0^{4/\beta} e^{\epsilon+\epsilon_{\text{max}}}\Theta(\epsilon_{\text{max}}-\epsilon)\left[1+e^{(\beta+1-\beta\gamma)(\epsilon_{\text{max}}-\epsilon)}((\beta+1-\beta\gamma)(\epsilon_{\text{max}}-\epsilon)-1)\right], \tag{S20}$$

with $\epsilon_{\text{max}} = \ln\left(\frac{N}{2\kappa_0^{2/\beta}}\right)$. Note that at $\beta = 1$ these two are the same. The general form is thus

$$\rho(\epsilon) = ae^{\epsilon+\epsilon_{\text{max}}}\Theta(\epsilon_{\text{max}}-\epsilon)\left[1+e^{b(\epsilon_{\text{max}}-\epsilon)}(b(\epsilon_{\text{max}}-\epsilon)-1)\right]. \tag{S21}$$

Supplementary Note 1.2.2 Chemical Potential

With this we can calculate the chemical potential. In order to do so we study the average amount of links

$$\begin{aligned}\langle E\rangle &= \int_{-\infty}^{\epsilon_{\text{max}}} \mathrm{d}\epsilon \frac{\rho(\epsilon)}{1+e^{\beta(\epsilon-\mu)}} \\ &= ae^{2\epsilon_{\text{max}}}\Bigg(\frac{b^2}{(b-1)^2} + e^{\beta(\epsilon_{\text{max}}-\mu)}\Bigg(-\frac{b\Phi\left[-e^{\beta(\epsilon_{\text{max}}-\mu)}, 2, \frac{1-b+\beta}{\beta}\right]}{\beta^2} - \frac{{}_2F_1\left[\begin{matrix}1, 1+\frac{1}{\beta}\\ 2+\frac{1}{\beta}\end{matrix}; -e^{\beta(\epsilon_{\text{max}}-\mu)}\right]}{1+\beta} \\ &\qquad + \frac{{}_2F_1\left[\begin{matrix}1, 1+\frac{1-b}{\beta}\\ 2+\frac{1-b}{\beta}\end{matrix}; -e^{\beta(\epsilon_{\text{max}}-\mu)}\right]}{1+\beta-b}\Bigg)\Bigg).\end{aligned} \tag{S22}$$

Here, $\Phi[z, a, b]$ is the Lerch zeta function. If we now assume $e^{\beta(\epsilon_{\text{max}}-\mu)} \gg 1$, we can approximate this as

$$\langle E\rangle \simeq ae^{(2+\beta)\epsilon_{\text{max}}-\beta\mu}\left\{\frac{1}{\beta}\pi\csc\left(\frac{\pi}{\beta}\right)e^{-(1+\beta)(\epsilon_{\text{max}}-\mu)} + \frac{b^2}{(1-\beta)(b+\beta-1)^2}e^{-2\beta(\epsilon_{\text{max}}-\mu)}\right\}. \tag{S23}$$

We know that $\epsilon_{\text{max}} \sim \ln N$ so $\mu \simeq c\ln N$ where $c < 1$. It can then be shown that for all c the dominant contributions are

$$\langle E\rangle \simeq \begin{cases} \frac{a\pi}{\beta}e^{\epsilon_{\text{max}}+\mu}\csc\left(\frac{\pi}{\beta}\right) & \text{if} \quad \beta > 1 \\ a\epsilon_{\text{max}}e^{\epsilon_{\text{max}}+\mu} & \text{if} \quad \beta = 1 \\ \frac{ab^2}{(1-\beta)(b+\beta-1)^2}e^{(2-\beta)\epsilon_{\text{max}}+\mu\beta} & \text{if} \quad \beta < 1 \end{cases} \tag{S24}$$

If we take $\langle E\rangle = N\langle k\rangle/2$ (sparse network) we obtain

$$\mu \simeq \begin{cases} \ln\left(\frac{\beta\sin\left(\frac{\pi}{\beta}\right)}{2\pi\langle k\rangle}\right) & \text{if } \beta > 1 \\ \frac{1}{2\langle k\rangle\ln N} & \text{if } \beta = 1 \\ \frac{1}{\beta}\ln\left(\frac{N^{\beta-1}(1-\beta)}{2^\beta\langle k\rangle}\right) & \text{if } \beta < 1 \end{cases} \tag{S25}$$

Note that in all these cases $e^{\beta(\epsilon_{\text{max}}-\mu)} \gg 1$ and that these are exactly the same results as we found before.

Supplementary Note 1.2.3 Thermodynamics

With this we can now study the grand potential.

$$\begin{aligned}
\Xi &= -\frac{1}{\beta}\int_{-\infty}^{\epsilon_{\max}} \mathrm{d}\epsilon\rho(\epsilon)\ln\left(1+e^{-\beta(\epsilon-\mu)}\right) \\
&= -\frac{a}{\beta}e^{2\epsilon_{\max}}\Bigg\{\frac{b}{\beta(1-b)}\Phi\left[-e^{\beta(\epsilon_{\max}-\mu)},2,\frac{1-b}{\beta}\right]+(-1)^{-1/\beta}e^{-(\epsilon_{\max}-\mu)}B_{-e^{\beta(\epsilon_{\max}-\mu)}}\left[1+1/\beta,0\right] \\
&\frac{1-2b+(b-1)b\epsilon_{\max}}{1-b+\beta}\frac{\beta}{(1-b)^2}{}_2F_1\left[\begin{array}{c}1,1+\frac{1-b}{\beta}\\ 2+\frac{1-b}{\beta}\end{array};-e^{\beta(\epsilon_{\max}-\mu)}\right]e^{\beta(\epsilon_{\max}-\mu)} \\
&+\frac{\beta b}{(b-1)^3}(1-\epsilon_{\max}+b(-3+b+\epsilon_{\max}))+\frac{b^2}{(1-b)^2}\ln\left(1+e^{-\beta(\epsilon_{\max}-\mu)}\right) \\
&-\frac{b\beta}{(1-b)^2}\epsilon_{\max}{}_2F_1\left[\begin{array}{c}1,\frac{1-b}{\beta}\\ 1+\frac{1-b}{\beta}\end{array};-e^{\beta(\epsilon_{\max}-\mu)}\right]e^{\beta(\epsilon_{\max}-\mu)}\Bigg\},
\end{aligned} \tag{S26}$$

where $B_z[a,b]$ is the incomplete beta function. Again assuming that $e^{\beta(\epsilon_{\max}-\mu)} \gg 1$ and $b<1$ we get the following dominant terms, after having divided out $\langle E\rangle$

$$\frac{\Xi}{E} \simeq \begin{cases} -1 & \text{if} \quad \beta>1 \\ -1 & \text{if} \quad \beta=1 \\ -\frac{1}{\beta} & \text{if} \quad \beta<1 \end{cases} \tag{S27}$$

Normally with this we have enough to calculate the entropy. However, we need to be careful when using a temperature dependent density of states. Let us check if $S=\beta^2\left(\frac{\partial\Xi}{\partial\beta}\right)_\mu$ still holds.

$$\begin{aligned}
\beta^2\left(\frac{\partial\Xi}{\partial\beta}\right)_\mu &= \underbrace{-\beta\rho(\epsilon_{\max})\frac{\partial\epsilon_{\max}}{\partial\beta}\ln\left(1+e^{-\beta(\epsilon_{\max}-\mu)}\right)-\beta\int_{-\infty}^{\epsilon_{\max}}\mathrm{d}\epsilon\frac{\partial\rho(\epsilon)}{\partial\beta}\ln\left(1+e^{-\beta(\epsilon_{\max}-\mu)}\right)}_{\Delta} \\
&+\underbrace{\int_{-\infty}^{\epsilon_{\max}}\mathrm{d}\epsilon\rho(\epsilon)\left(\ln\left(1+e^{-\beta(\epsilon_{\max}-\mu)}\right)+\beta(\epsilon-\mu)\frac{1}{1+\epsilon^{\beta(\epsilon-\mu)}}\right)}_{-\beta\Xi+\beta(\langle U\rangle-\mu\langle E\rangle)} \\
&= \Delta+\int_{-\infty}^{\epsilon_{\max}}\mathrm{d}\epsilon\rho(\epsilon)\left(\frac{\ln\left(1+e^{\beta(\epsilon_{\max}-\mu)}\right)}{1+\epsilon^{\beta(\epsilon-\mu)}}+\frac{\ln\left(1+e^{-\beta(\epsilon_{\max}-\mu)}\right)}{1+\epsilon^{-\beta(\epsilon-\mu)}}\right) \\
&= \Delta-\int_{-\infty}^{\epsilon_{\max}}\mathrm{d}\epsilon\rho(\epsilon)\left(\underbrace{\frac{1}{1+\epsilon^{\beta(\epsilon-\mu)}}}_{p(\epsilon)}\ln\left(\frac{1}{1+e^{\beta(\epsilon_{\max}-\mu)}}\right)+\left(1-\frac{1}{1+\epsilon^{\beta(\epsilon-\mu)}}\right)\ln\left(1-\frac{1}{1+e^{\beta(\epsilon_{\max}-\mu)}}\right)\right) \\
&= \Delta-\int_{-\infty}^{\epsilon_{\max}}\mathrm{d}\epsilon\rho(\epsilon)\Big(p(\epsilon)\ln(p(\epsilon))+(1-p(\epsilon))\ln(1-p(\epsilon))\Big)=\Delta+S
\end{aligned} \tag{S28}$$

In the last step we recognize the entropy of a graphon gas [5]. So, indeed, in the case that $\rho(\epsilon)$ or $\epsilon_{\max}$ depends on the temperature, we get extra terms ($S=\beta^2\left(\frac{\partial\Xi}{\partial\beta}\right)_\mu-\Delta$). These terms, at least in the general case, are not trivial to evaluate.

However, we also note that $S = \beta(\langle U\rangle - \Xi - \mu\langle E\rangle)$ remains valid in all cases. We will therefore approach S in this way. Thus, the final thing we need to do is find an expression for the average energy.

$$\begin{aligned}
\langle U\rangle &= \int_{-\infty}^{\epsilon_{\max}} \mathrm{d}\epsilon \frac{\epsilon\rho(\epsilon)}{1+e^{\beta(\epsilon-\mu)}} \\
&= ae^{2\epsilon_{\max}}\Bigg\{ {}_2F_1\left[\begin{array}{c} 1, \frac{1-b}{\beta} \\ 1+\frac{1-b}{\beta} \end{array}; -e^{\beta(\epsilon_{\max}-\mu)}\right]\epsilon_{\max} + \frac{1}{b-1}{}_2F_1\left[\begin{array}{c} 1, \frac{1-b}{\beta} \\ 1+\frac{1-b}{\beta} \end{array}; -e^{\beta(\epsilon_{\max}-\mu)}\right]\epsilon_{\max} \\
&+ \frac{1+b\epsilon_{\max}}{(b-1)^2}{}_3F_2\left[\begin{array}{c} 1, \frac{1-b}{\beta}, \frac{1-b}{\beta} \\ 1+\frac{1-b}{\beta}, 1+\frac{1-b}{\beta} \end{array}; -e^{\beta(\epsilon_{\max}-\mu)}\right] - {}_3F_2\left[\begin{array}{c} 1, \frac{1}{\beta}, \frac{1}{\beta} \\ 1+\frac{1}{\beta}, 1+\frac{1}{\beta} \end{array}; -e^{\beta(\epsilon_{\max}-\mu)}\right] \\
&+ \frac{2b}{(b-1)^2}{}_4F_3\left[\begin{array}{c} 1, \frac{1-b}{\beta}, \frac{1-b}{\beta}, \frac{1-b}{\beta} \\ 1+\frac{1-b}{\beta}, 1+\frac{1-b}{\beta}, 1+\frac{1-b}{\beta} \end{array}; -e^{\beta(\epsilon_{\max}-\mu)}\right]\Bigg\}
\end{aligned} \tag{S29}$$

We can again take the limit $e^{\beta(\epsilon_{\max}-\mu)} \gg 1$, dividing out $\langle E\rangle$, to obtain

$$\frac{\langle U\rangle}{\langle E\rangle} \simeq \begin{cases} \mu - \frac{\pi}{\beta}\cot\left(\frac{\pi}{\beta}\right) & \text{if} \quad \beta > 1 \\ \frac{1}{b} + \frac{1}{2}\epsilon_{\max} + \frac{1}{2}\mu & \text{if} \quad \beta = 1 \\ \epsilon_{\max} - \frac{b+3\beta-3}{(1-\beta)(b+\beta-1)} & \text{if} \quad \beta < 1. \end{cases} \tag{S30}$$

Finally, this leads us to the entropy:

$$\frac{S}{\langle E\rangle} = \beta\left(\frac{\langle U\rangle}{\langle E\rangle} - \frac{\Xi}{\langle E\rangle} - \mu\right) \simeq \begin{cases} \beta\left(\mu - \frac{\pi}{\beta}\cot\left(\frac{\pi}{\beta}\right) + 1 - \mu\right) & \text{if} \quad \beta > 1 \\ \frac{1}{b} + \frac{1}{2}\epsilon_{\max} + \frac{1}{2}\mu + 1 - \mu & \text{if} \quad \beta = 1 \\ \beta\left(\epsilon_{\max} - \frac{b+3\beta-3}{(1-\beta)(b+\beta-1)} + \frac{1}{\beta} - \mu\right) & \text{if} \quad \beta < 1 \end{cases} \tag{S31}$$

Now we plug in the remaining variables to obtain

$$\frac{S}{\langle E\rangle} \simeq \begin{cases} \beta - \pi\cot\left(\frac{\pi}{\beta}\right) & \text{if } \beta > 1 \\ 1 + \frac{1}{2}\ln N - \frac{1}{2}\ln\langle k\rangle + \ln\left(\frac{\gamma-1}{\gamma-2}\right) + \frac{1}{2-\gamma} + \frac{1}{2}\ln\ln N & \text{if } \beta = 1 \\ 1 + \ln N - \ln\langle k\rangle - \ln(1-\beta) + \frac{\beta}{\beta-1} + 2\ln\left(\frac{\gamma-1}{\gamma-2}\right) - \frac{2}{\gamma-2} & \text{if } \beta < 1 \end{cases} \tag{S32}$$

The final entropy is, as expected, equal to that of the an Erdös-Renyi graph with connection probability $\langle k\rangle/N$ when $\beta \to 0$ and $\gamma \to \infty$[6] and should give the entropy of the soft configuration model when $\beta \to 0$.

Using the density of states and the Fermi-Dirac statistics we can also find the probability of a link having energy ϵ. This is namely given by

$$p(\epsilon) = \frac{1}{\langle E\rangle}\frac{g(\epsilon)}{1+e^{\beta(\epsilon-\mu)}} \tag{S33}$$

We can plug in the approximated values of μ from before and plot the results for $\beta = 1/2$ (blue line) and $\beta = 3/2$ (orange line) in Supplementary Figure 3. Notice that this probability density changes dramatically at the "critical" point $\beta = 1$. Indeed, when $\beta > 1$ particles occupy low energy states and for $\beta < 1$ they occupy mainly high energy states. However, since the number of states grows exponentially with the energy, the number of available microstates per particle grows extremely fast in the $\beta < 1$ regime, inducing a sudden increase of the entropy, explaining the divergence of the entropy in the thermodynamic limit in this regime.

Supplementary Note 1.2.4 Toy model

Above we have seen some interesting bahavior, most notably the non-extensivity of the entropy above the critical temperature. We want to now investigate where this feature comes from, by looking at a simplified version of our model. Suppose we have a system made of N_{part} non-interacting "particles", each of which can attain states of energy $\epsilon \in (0, \epsilon_{\text{max}})$. Suppose also that the degeneracy of states of energy ϵ grows as

$$g(\epsilon) = V e^{\beta_c \epsilon}$$

with β_c a fixed parameter and V the volume of the system. The probability density to find one such particle in a state of energy ϵ is

$$p(\epsilon) = \frac{\beta - \beta_c}{1 - e^{-(\beta-\beta_c)\epsilon_{\text{max}}}} e^{-(\beta-\beta_c)\epsilon}. \tag{S34}$$

We notice that here we find the same sudden change of behavior at the critical point $\beta = \beta_c$ as we found in the $\mathbb{S}^1$ model. Using Maxwell-Boltzmann statistics for identical particles, the entropy per particle of this system is easily calculated as

$$\frac{S}{N_{\text{part}}} = \frac{\beta}{\beta - \beta_c} - \frac{\beta \epsilon_{\text{max}}}{e^{(\beta-\beta_c)\epsilon_{\text{max}}} - 1} - \ln\left[\frac{N_{\text{part}}}{V} \frac{\beta - \beta_c}{1 - e^{-(\beta-\beta_c)\epsilon_{\text{max}}}}\right] + 1. \tag{S35}$$

The first two terms in this last equation are just the average energy per particle of the system. If the density of particles is kept fixed, so that $\lim_{N_{\text{part}}\to\infty} \frac{N_{\text{part}}}{V} = \text{cte}$, then entropy is an extensive quantity as it is proportional to the number of particles. However, there is a clear change of behavior as one goes from $\beta > \beta_c$ to $\beta < \beta_c$ due to the change of behavior of the probability density Eq. (S34). If besides $\beta\epsilon_{\text{max}} \gg 1$, then the entropy behaves as

$$\frac{S}{N_{\text{part}}} \simeq \begin{cases} \frac{\beta}{\beta-\beta_c} - \ln\left[\frac{N_{\text{part}}}{V}(\beta - \beta_c)\right] + 1 & \beta > \beta_c \\ \frac{1}{2}\beta_c \epsilon_{\text{max}} - \ln\left[\frac{N_{\text{part}}}{V\epsilon_{\text{max}}}\right] + 1 & \beta = \beta_c \\ \beta_c \epsilon_{\text{max}} + \frac{\beta}{\beta-\beta_c} - \ln\left[\frac{N_{\text{part}}}{V}(\beta_c - \beta)\right] + 1 & \beta < \beta_c \end{cases} \tag{S36}$$

Thus, in the limit of $\epsilon_{\text{max}} \to \infty$ the entropy per particle diverges at $\beta \to \beta_c^+$ and scales as ϵ_{max} for $\beta \leq \beta_c$, just as in our model.

In the $\mathbb{S}^1$-model, the effective system size is given by the proportionality constant $V_{\text{eff}} = ae^{\epsilon_{\text{max}}}$ in Eq. (S21) and the amount of particles is given by $\langle E \rangle = N\langle k \rangle/2$ as we are working with a sparse graph. In this case we indeed satisfy $\lim_{\langle E \rangle \to \infty} \frac{\langle E \rangle}{V_{\text{eff}}} = \text{cte}$ and the entropy is thus in principle extensive. However, as in the full model there is the extra constraint $\langle E \rangle \leq N(N-1)/2$ and $\epsilon_{\text{max}} = \ln(N/(2\kappa_0^2))$, we are obliged to also send ϵ_{max} to infinity when going to the thermodynamic limit, thus resulting in a non-extensive entropy for $\beta < \beta_c$. We thus show that the essential feature of the $\mathbb{S}^1$ model that leads to a non-extensive entropy is the exponential dependence on the energy of the density of states.

Supplementary Note 1.3 Scaling Behaviour of Clustering with System Size

In the following section we find the dominant finite size scaling of the clustering coefficient for $\beta \leq 1$. As was explained in the main text, in this region in the thermodynamic limit clustering vanishes. We will therefore study what happens when $N \gg 1$ but finite for any β (we thus do not take any limit with respect to the temperature). As for $\beta \lesssim 1$ higher order finite size correction become important, we study separately the case $\beta = 1$.

We start by manipulating the angular integrals of Eq. (S1) as to simplify the task at hand later on. We then turn to the scaling when $\beta < 1$ and conclude with an analysis of the scaling when $\beta = 1$. In the case of $\beta < 1$, in order to facilitate numerics later on, we choose to adopt the connection probability as defined by Eq. (S12), where the degree sequence at different temperatures is the same.

The basis of these calculations is the fact that we are looking for the scaling behaviour of the $\overline{c}$ with respect to the system size N. This allows us to always ignore terms that we know are smaller than than the main term, which simplifies the integrals that we study substantially. Once we have a term, say A, we want to know the scaling behaviour of, we use the fact that if the functions $f(N)$ and $g(N)$ in equation

$$f(N) < A < g(N) \tag{S37}$$

have the same dominant scaling, one can immediately conclude that A also has that exact dominant scaling. Therefore, by

finding upper and lower bounds to the integrals in question we can extract there scaling behavior with respect to A. It is important to keep in mind that, even when the integrals representing the bounds become very tedious, the strategy we employ remains the same throughout this section.

Supplementary Note 1.3.1 Angular Manipulation

We start by manipulating the angular integrals of Eq. (S1) to make it easier to work with, i.e. get rid of the absolute values in the expressions for $\Delta\theta$. The equation has the following form:

$$\overline{c}(\kappa) = \frac{\iiiint \mathrm{d}\kappa' \mathrm{d}\kappa'' \mathrm{d}\theta' \mathrm{d}\theta'' \rho(\kappa')\rho(\kappa'') p(\kappa, \kappa', \pi - |\pi - |\theta'||) p(\kappa, \kappa'', \pi - |\pi - |\theta''||) p(\kappa', \kappa'', \pi - |\pi - |\theta' - \theta''||))}{\iint \mathrm{d}\kappa' \mathrm{d}\theta' \rho(\kappa') p(\kappa, \kappa', \pi - |\pi - |\theta'||)}. \tag{S38}$$

Here, we have used $\theta = 0$. Let us first investigate the trivial case of the denominator, where we only focus on the angular integral

$$\begin{aligned}\int_0^{2\pi} \mathrm{d}\theta' p(\kappa, \kappa', \pi - |\pi - |\theta'||) &= \int_0^{\pi} \mathrm{d}\theta' p(\kappa, \kappa', \pi - |\pi - |\theta'||) + \int_{\pi}^{2\pi} \mathrm{d}\theta' p(\kappa, \kappa', \pi - |\pi - |\theta'||) \\ &= \int_0^{\pi} \mathrm{d}\theta' p(\kappa, \kappa', \theta') + \int_{\pi}^{2\pi} \mathrm{d}\theta' p(\kappa, \kappa', 2\pi - \theta') = 2\int_0^{\pi} \mathrm{d}\theta' p(\kappa, \kappa', \theta'),\end{aligned} \tag{S39}$$

where in the last step we have performed the transformation $t = 2\pi - \theta'$ and $t \to \theta'$ on the second integral. The numerator can be rewritten in a similar way to obtain four terms

$$\begin{aligned}&\int_0^{2\pi} \mathrm{d}\theta' \int_0^{2\pi} \mathrm{d}\theta'' p(\kappa, \kappa', \pi - |\pi - |\theta'||) p(\kappa, \kappa'', \pi - |\pi - |\theta''||) p(\kappa', \kappa'', \pi - |\pi - |\theta' - \theta''||) \\ =&2\int_0^{\pi} \mathrm{d}\theta' \Bigg(\int_0^{\theta'} \mathrm{d}\theta'' p(\kappa, \kappa', \theta') p(\kappa, \kappa'', \theta'') p(\kappa', \kappa'', \theta' - \theta'') + \int_0^{\theta'} \mathrm{d}\theta'' p(\kappa, \kappa', \theta'') p(\kappa, \kappa'', \theta') p(\kappa', \kappa'', \theta' - \theta'') \\ &+ \int_0^{\pi - \theta'} \mathrm{d}\theta'' p(\kappa, \kappa', \theta') p(\kappa, \kappa'', \theta'') p(\kappa', \kappa'', \theta' + \theta'') + \int_{\pi - \theta'}^{\pi} \mathrm{d}\theta'' p(\kappa, \kappa', \theta') p(\kappa, \kappa'', \theta'') p(\kappa', \kappa'', 2\pi - \theta' - \theta'') \Bigg).\end{aligned} \tag{S40}$$

The first two terms are not exactly the same. However, as the full expression of the clustering coefficient also contains integrals over the hidden degrees, one can interchange $\kappa' \leftrightarrow \kappa''$. This thus shows that the first two terms contribute equally to the clustering coefficient. All in all, we will thus be working with the following three terms

$$\begin{aligned}&4\int_0^{\pi} \mathrm{d}\theta' \int_0^{\theta'} \mathrm{d}\theta'' p(\kappa, \kappa', \theta') p(\kappa, \kappa'', \theta'') p(\kappa', \kappa'', \theta' - \theta'') \\ +&2\int_0^{\pi} \mathrm{d}\theta' \int_0^{\pi - \theta'} \mathrm{d}\theta'' p(\kappa, \kappa', \theta') p(\kappa, \kappa'', \theta'') p(\kappa', \kappa'', \theta' + \theta'') \\ +&2\int_0^{\pi} \mathrm{d}\theta' \int_{\pi - \theta'}^{\pi} \mathrm{d}\theta'' p(\kappa, \kappa', \theta') p(\kappa, \kappa'', \theta'') p(\kappa', \kappa'', 2\pi - \theta' - \theta'').\end{aligned} \tag{S41}$$

Now, before we get started on finding the scaling with respect to the system size of each term individually, it might be that we can avoid doing so by some simple arguments. Indeed, we will show that the first term will always dominate the others in the large N limit, and so we only have to find its scaling. Let us start with the second term

$$\begin{aligned}
&2\iint \mathrm{d}\kappa'\mathrm{d}\kappa''\rho(\kappa')\rho(\kappa'')\int_0^\pi \mathrm{d}\theta'\int_0^{\pi-\theta'}\mathrm{d}\theta'' p(\kappa,\kappa',\theta')p(\kappa,\kappa'',\theta'')p(\kappa',\kappa'',\theta'+\theta'')\\
\leq &2\iint \mathrm{d}\kappa'\mathrm{d}\kappa''\rho(\kappa')\rho(\kappa'')\int_0^\pi \mathrm{d}\theta'\int_0^{\pi}\mathrm{d}\theta'' p(\kappa,\kappa',\theta')p(\kappa,\kappa'',\theta'')p(\kappa',\kappa'',\theta'+\theta'').
\end{aligned} \tag{S42}$$

The above statement is true as the integrand is strictly positive and so extending the integration domain will only make the integral larger. Now, we can split the θ'' integral and perform $\theta' \leftrightarrow \theta''$ and $\kappa' \leftrightarrow \kappa''$ on the second term to obtain

$$\begin{aligned}
&2\iint \mathrm{d}\kappa'\mathrm{d}\kappa''\rho(\kappa')\rho(\kappa'')\int_0^\pi \mathrm{d}\theta'\int_0^{\pi}\mathrm{d}\theta'' p(\kappa,\kappa',\theta')p(\kappa,\kappa'',\theta'')p(\kappa',\kappa'',\theta'+\theta'')\\
&\qquad = 4\iint \mathrm{d}\kappa'\mathrm{d}\kappa''\rho(\kappa')\rho(\kappa'')\int_0^\pi \mathrm{d}\theta'\int_0^{\theta'}\mathrm{d}\theta'' p(\kappa,\kappa',\theta')p(\kappa,\kappa'',\theta'')p(\kappa',\kappa'',\theta'+\theta'')\\
&\qquad \leq 4\iint \mathrm{d}\kappa'\mathrm{d}\kappa''\rho(\kappa')\rho(\kappa'')\int_0^\pi \mathrm{d}\theta'\int_0^{\theta'}\mathrm{d}\theta'' p(\kappa,\kappa',\theta')p(\kappa,\kappa'',\theta'')p(\kappa',\kappa'',\theta'-\theta'').
\end{aligned} \tag{S43}$$

In the final step we use the functional form of p with respect to the angular coordinate is

$$p(s) = \frac{1}{1+s^\beta}. \tag{S44}$$

As s^β is monotonously increasing, and $1/(1+s)$ is monotonously decreasing, $p(s)$ is monotonously decreasing. Thus, it is largest when s is smallest. Obviously, $\theta'+\theta'' > \theta'-\theta''$ for all $(\theta',\theta'') \in [0,\pi]\times[0,\theta']$. We have thus proven that the first term in Eq. (S41) dominates the second term. We can follow similar steps for the third term. We we will now only clarify steps if they are new.

$$\begin{aligned}
&2\iint\limits_{\kappa',\kappa''} \mathrm{d}\kappa'\mathrm{d}\kappa''\rho(\kappa')\rho(\kappa'')\int\limits_0^\pi \mathrm{d}\theta'\int\limits_{\pi-\theta'}^{\pi}\mathrm{d}\theta'' p(\kappa,\kappa',\theta')p(\kappa,\kappa'',\theta'')p(\kappa',\kappa'',2\pi-\theta'-\theta'')\\
\leq &4\iint\limits_{\kappa',\kappa''} \mathrm{d}\kappa'\mathrm{d}\kappa''\rho(\kappa')\rho(\kappa'')\int\limits_0^\pi \mathrm{d}\theta'\int\limits_{0}^{\theta'}\mathrm{d}\theta'' p(\kappa,\kappa',\theta')p(\kappa,\kappa'',\theta'')p(\kappa',\kappa'',2\pi-\theta'-\theta'').
\end{aligned} \tag{S45}$$

Now, one knows that $2\pi-\theta'-\theta'' \geq \theta'-\theta''\ \forall_{(\theta',\theta'')\in[0,\pi]\times[0,\theta']}$. For the same reasons as before, this then implies

$$\begin{aligned}
&4\iint\limits_{\kappa',\kappa''} \mathrm{d}\kappa'\mathrm{d}\kappa''\rho(\kappa')\rho(\kappa'')\int\limits_0^\pi \mathrm{d}\theta'\int\limits_{0}^{\theta'}\mathrm{d}\theta'' p(\kappa,\kappa',\theta')p(\kappa,\kappa'',\theta'')p(\kappa',\kappa'',2\pi-\theta'-\theta'')\\
\leq &4\iint\limits_{\kappa',\kappa''} \mathrm{d}\kappa'\mathrm{d}\kappa''\rho(\kappa')\rho(\kappa'')\int\limits_0^\pi \mathrm{d}\theta'\int\limits_{0}^{\theta'}\mathrm{d}\theta'' p(\kappa,\kappa',\theta')p(\kappa,\kappa'',\theta'')p(\kappa',\kappa'',\theta'-\theta''),
\end{aligned} \tag{S46}$$

so this term is also dominated by the first term in Eq. (S41).

Supplementary Note 1.3.2 Case $0 < \beta < 1$

The first step is to perform the transformation $x = \frac{\kappa'}{\kappa_s}$ and $y = \frac{\kappa''}{\kappa_s}$, where we define $\kappa_s^2 \equiv N^\beta/((2\pi)^\beta \hat{\mu})$. Note that we use assume the functional form of $\hat{\mu}$ defined in Eq. (S13), such that $\kappa_s \sim \sqrt{N}$. This leads to

$$\overline{c}(\kappa) \sim 2\,\frac{\int\limits_{\kappa_0/\kappa_s}^{\kappa_c/\kappa_s} \mathrm{d}x \int\limits_{\kappa_0/\kappa_s}^{\kappa_c/\kappa_s} \mathrm{d}y \int\limits_0^\pi \mathrm{d}\theta' \int\limits_0^{\theta'} \mathrm{d}\theta''(xy)^{-\gamma} p(\kappa,\kappa_s x,\theta')p(\kappa,\kappa_s y,\theta'')p(\kappa_s x,\kappa_s y,\theta'-\theta'')}{\left(\int\limits_{\kappa_0/\kappa_s}^{\kappa_c/\kappa_s} \mathrm{d}x \int\limits_0^\pi \mathrm{d}\theta' x^{-\gamma} p(\kappa,\kappa_s x,\theta')\right)^2}, \tag{S47}$$

We investigate the numerator and denominator separately and define

$$A_- = \int\limits_{\kappa_0/\kappa_s}^{\kappa_c/\kappa_s} \mathrm{d}x \int\limits_{\kappa_0/\kappa_s}^{\kappa_c/\kappa_s} \mathrm{d}y \int\limits_0^\pi \mathrm{d}\theta' \int\limits_0^{\theta'} \mathrm{d}\theta''(xy)^{-\gamma} p(\kappa,\kappa_s x,\theta')p(\kappa,\kappa_s y,\theta'')p(\kappa_s x,\kappa_s y,\theta'-\theta''). \tag{S48}$$

$$B = \int\limits_{\kappa_0/\kappa_s}^{\kappa_c/\kappa_s} \mathrm{d}x \int\limits_0^\pi \mathrm{d}\theta' x^{-\gamma} p(\kappa,\kappa_s x,\theta'). \tag{S49}$$

It is also useful to define

$$A_+ = \int\limits_{\kappa_0/\kappa_s}^{\kappa_c/\kappa_s} \mathrm{d}x \int\limits_{\kappa_0/\kappa_s}^{\kappa_c/\kappa_s} \mathrm{d}y \int\limits_0^\pi \mathrm{d}\theta' \int\limits_0^{\theta'} \mathrm{d}\theta''(xy)^{-\gamma} p(\kappa,\kappa_s x,\theta')p(\kappa,\kappa_s y,\theta'')p(\kappa_s x,\kappa_s y,\theta'+\theta''). \tag{S50}$$

Our investigation will focus on finding upper and lower bounds for these integrals. Note that from here on out we will drop the domains of the x and y integrals and assume them to be $[\kappa_0/\kappa_s, \kappa_c/\kappa_s]$ unless otherwise indicated. Using the fact that

$$\frac{1}{1+\dfrac{(\theta'+\theta'')^\beta}{xy}} < \frac{1}{1+\dfrac{(\theta'-\theta'')^\beta}{xy}}, \quad \forall_{\theta',\theta'',x,y}, \tag{S51}$$

we can conclude that $A_+ < A_-$. As numerical investigation leads us to expect that both have the same scaling, this implies that we do not need to worry about an upper bound for $A+$ nor the lower bound for A_-. If the functions $f(N)$ and $g(N)$ in equation

$$f(N) < A_+ < A_- < g(N) \tag{S52}$$

have the same dominant scaling, one can immediately conclude that A_- also has that exact dominant scaling. One might ask why we introduce A_+ in the first place, when in the end we are only interested in the scaling of A_-. The answer to this is that A_+ in general has nicer properties due to the lack of $(\theta'-\theta'')$, as it is thus easier to find a lower bound for it than for A_-.

We start with the simplest integral, the B-term, which can be solved exactly. To this end we first need to rewrite it a bit. By performing two substitutions

$$x' = \frac{\kappa_s}{\kappa_c} x \qquad x' \to x, \qquad t = \frac{\theta'}{\pi} \qquad t \to \theta', \tag{S53}$$

one obtains

$$B = \pi \left(\frac{\kappa_c}{\kappa_s}\right)^{1-\gamma} \int_0^1 \mathrm{d}\theta' \int_{\kappa_0/\kappa_c}^1 \mathrm{d}x \frac{x^{-\gamma}}{1 + \frac{(\pi\theta')^\beta \kappa_s^2}{\kappa_c \kappa x}}. \tag{S54}$$

This then gives the following expression

$$\begin{aligned} B = \frac{\pi}{(\beta(\gamma-1)-1)(\gamma-1)} \Bigg\{ & (\gamma-1)\beta \left(\frac{\kappa_0}{\kappa_s}\right)^{1-\gamma} {}_2F_1 \left[\begin{array}{c} 1, 1/\beta \\ 1+1/\beta \end{array} ; -\frac{\pi^\beta \kappa_s^2}{\kappa\kappa_0} \right] \\ & -(\gamma-1)\beta \left(\frac{\kappa_c}{\kappa_s}\right)^{1-\gamma} {}_2F_1 \left[\begin{array}{c} 1, 1/\beta \\ 1+1/\beta \end{array} ; -\frac{\pi^\beta \kappa_s^2}{\kappa\kappa_0} \right] \\ & - \quad \kappa_0 \quad \left(\frac{\kappa_0}{\kappa_s}\right)^{1-\gamma} {}_2F_1 \left[\begin{array}{c} 1, \gamma-1 \\ \gamma \end{array} ; -\frac{\pi^\beta \kappa_s^2}{\kappa\kappa_0} \right] \\ & + \quad \kappa_0 \quad \left(\frac{\kappa_c}{\kappa_s}\right)^{1-\gamma} {}_2F_1 \left[\begin{array}{c} 1, \gamma-1 \\ \gamma \end{array} ; -\frac{\pi^\beta \kappa_s^2}{\kappa\kappa_0} \right] \Bigg\}. \end{aligned} \tag{S55}$$

This expression can then be expanded w.r.t. N, using that $\kappa_s \sim \sqrt{N}$. To lowest order, one finds that B then scales as

$$B \sim N^{\frac{\gamma-3}{2}} \tag{S56}$$

Next we turn to the A_+ term. Here we use the following fact to bound this integral. If $F = \int_{\mathcal{V}} f(\vec{x})$, where $\mathcal{V}$ is the volume over which to integrate the function, and $f(\vec{x}) \geq f_0$ for all $\vec{x}$, where f_0 some constant, then $F \geq f_0 \mathcal{V}$. From the form of the standard connection probability given in Eq. (S44), we see that A_+ is smallest when the argument is largest, which is the case when θ', θ'' are largest, so when they are both π. Thus we can bound the angular integrals by replacing the integrand with its minimum, the same function where both angular coordinates are π. The integrand is then a constant so the bound is given by the value of that constant times the area of the integral. Plugging this in we obtain

$$\begin{aligned} A_+ &\geq \frac{1}{2}\pi^2 \int \mathrm{d}x\mathrm{d}y (xy)^{-\gamma} \frac{1}{1 + \frac{\pi^\beta \kappa_s}{\kappa x}} \frac{1}{1 + \frac{\pi^\beta \kappa_s}{\kappa y}} \frac{1}{1 + \frac{(2\pi)^\beta}{xy}} \\ &= \frac{1}{2}\pi^{2-3\beta} \left(\frac{\kappa}{\kappa_s}\right)^2 \int \mathrm{d}x\mathrm{d}y (xy)^{2-\gamma} \frac{1}{1 + \frac{\kappa x}{\pi^\beta \kappa_s}} \frac{1}{1 + \frac{\kappa y}{\pi^\beta \kappa_s}} \frac{1}{1 + \frac{xy}{(2\pi)^\beta}}. \end{aligned} \tag{S57}$$

Now, this is exactly the same integral (with the exception of the π's, but they will obviously not change scaling) as the one evaluated in Ref. [3]. As was found in the reference (Eq. (6)), the scaling depends on how we set κ_c relative to κ_s. We

distinguish two regimes. First, there is the regime where $\kappa_0 \ll \kappa_s \ll \kappa_c$. In this case, the scaling is

$$A_+ \geq c_{+,1}\kappa_s^{-2}\ln(\kappa_c/\kappa_s). \tag{S58}$$

Then, there is the region where $\kappa_0 \leq \kappa_c \leq \kappa_s$ ($\kappa_0 \ll \kappa_s$ must be required to hold) where one obtains

$$A_+ \geq c_{+,2}\kappa_s^{2\gamma-8}\kappa_c^{6-2\gamma}. \tag{S59}$$

This, however, does not give the full scaling behaviour, as numerical results show us that for large β the scaling with respect to N is different. To find where this different scaling comes from we take a step back and look at the full integral A_+ as given in Eq. (S50). One might be tempted to, as in Ref. [3], expand the first two connection probabilities to first order. However, the presence of the angular coordinate makes this impossible. The argument of these connection probabilities has the form $s = \frac{\theta^\beta \kappa_s^2}{\kappa\kappa'}$. It becomes clear that for small enough θ, s is no longer large and the approximation thus breaks down. We thus expect different scaling behaviour to arise as a result of small angular coordinates. To investigate this further, we split the *angular* integration domain $[0,\pi]\times[0,t]$ in a convenient way and investigate the domain $\mathcal{D}_1 = [0,(xy)^{1/\beta}]\times[0,t]$. Note that we do not have to look at the other half of the original domain as we are only interested in the lower bound and our integrand is positive for all angles, which means that the integral over the full domain must be larger or equal to the integral over D_1. The domain $\mathcal{D}_1$ can only be defined in the case that $\kappa_c \leq \kappa_s$, as only then the angular coordinates remain smaller than the maximal possible value of π for all x and y. For the case that $\kappa_c \gg \kappa_s$ we define the more restrictive domain $\mathcal{D}_2 = [0,(\kappa_0/\kappa_s)^{2/\beta}]\times[0,t]$. Starting with the case $\kappa_c \leq \kappa_s$, bounding the integral as before (by replacing the integrand by its minimum), one finds

$$\begin{aligned}
A_+ &\geq \frac{1}{1+2^\beta}\int \mathrm{d}x\mathrm{d}y (xy)^{2/\beta-\gamma}\frac{1}{1+\frac{\kappa_s y}{\kappa}}\frac{1}{1+\frac{\kappa_s x}{\kappa}} \\
&= \frac{(\kappa_s/\kappa)^{-4/\beta+2\gamma-2}}{1+2^\beta}\left(B_{\frac{\kappa}{\kappa_0+\kappa}}\left[\gamma-\frac{2}{\beta},1-\gamma+\frac{2}{\beta}\right]-B_{\frac{\kappa}{\kappa_c+\kappa}}\left[\gamma-\frac{2}{\beta},1-\gamma+\frac{2}{\beta}\right]\right)^2 \\
&\simeq c_{+,s,1}\kappa_s^{-4/\beta+2\gamma-2}+c_{+,s,2}\kappa_s^{-4/\beta+2\gamma-2}\kappa_c^{4/\beta-2\gamma},
\end{aligned} \tag{S60}$$

For the case $\kappa_c \gg \kappa_s$ one obtains

$$A_+ \geq \left(\frac{\kappa_0}{\kappa_s}\right)^{4/\beta}\int \mathrm{d}x\mathrm{d}y (xy)^{-\gamma}\frac{1}{1+\frac{\kappa_s}{\kappa x}\frac{\kappa_0^2}{\kappa_s^2}}\frac{1}{1+\frac{\kappa_s}{\kappa y}\frac{\kappa_0^2}{\kappa_s^2}}\frac{1}{1+\frac{2^\beta}{xy}\frac{\kappa_0^2}{\kappa_s^2}}$$

$$\simeq \left(\frac{\kappa_0}{\kappa_s}\right)^{4/\beta} \int \mathrm{d}x\mathrm{d}y (xy)^{-\gamma} \simeq c_{+,s,3}\kappa_s^{-4/\beta+2\gamma-2}, \tag{S61}$$

where in the first step it was noted that irrespective of the value of x and y, the argument of the connection probabilities is small.

We now have five different scaling behaviours. Which terms dominate will depend on the value of β as well on κ_c. To quantify how the scaling varies with κ_c we introduce the exponent α such that $\kappa_c \sim N^{\alpha/2}$. As $\kappa_s \sim N^{1/2}$, the different regimes of κ_c described above correspond to $\alpha \in [0,1]$ for $\kappa_c \leq \kappa_s$ and $\alpha \in (1, \frac{2}{\gamma-1}]$ for $\kappa_c \gg \kappa_s$. Using these definitions and adding up the different scaling we found above, we conclude that

$$A_+ \geq \begin{cases} C_{+,1}N^{-2/\beta+\gamma-1} + C_{+,2}N^{-1}\ln N & \text{if} \quad \kappa_c \gg \kappa_s \\ N^{-1}\left(C_{+,3}N^{\gamma-2/\beta} + C_{+,4}N^{(1-\alpha)(\gamma-2/\beta)} + C_{+,5}N^{(1-\alpha)(\gamma-3)}\right) & \text{if} \quad \kappa_c \leq \kappa_s \end{cases}, \tag{S62}$$

where $C_{+,i}$ are constants. Note that, for example, the scaling of Eqs. (S58) and (S61) can indeed be combined to the first of these two inequalities as both now hold for all β. When $\beta > 2/\gamma$ the $C_{+,2}$-term vanished with respect to the $C_{+,1}$-term and we are left with inequality (S61) and when $\beta < 2/\gamma$ the other term dominates and we are left with inequality (S58).

Now obviously this is just a lower bound. To show that the clustering indeed scales like this we must also find an upper bound, which we do by turning to the A_- term. We divide the integration domain in two: $\mathcal{D}_s = [0, (\kappa_0/\kappa_s)^{2/\beta}] \times [0, \theta']$ and $\mathcal{D}_l = [(\kappa_0/\kappa_s)^{2/\beta}, \pi] \times [0, \theta']$. We first turn to region $\mathcal{D}_l$.

$$\begin{aligned}
A_{-,l} &= \iint_{\mathcal{D}_l} \mathrm{d}\theta'\mathrm{d}\theta'' \iint \mathrm{d}x\mathrm{d}y (xy)^{-\gamma} \quad \frac{1}{1+\frac{\theta'^{\beta}\kappa_s}{\kappa x}} \frac{1}{1+\frac{\theta''^{\beta}\kappa_s}{\kappa y}} \frac{1}{1+\frac{(\theta'-\theta'')^{\beta}}{xy}} \\
&\leq \left(\frac{\kappa}{\kappa_s}\right)^2 \iint_{\mathcal{D}_l} \mathrm{d}\theta'\mathrm{d}\theta'' \iint \mathrm{d}x\mathrm{d}y \;\; (xy)^{2-\gamma}(\theta'\theta''(\theta'-\theta''))^{-\beta} \frac{1}{1+\frac{xy}{(\theta'-\theta'')^{\beta}}} \\
&= \left(\frac{\kappa}{\kappa_s}\right)^2 \iint_{\mathcal{D}_l} \frac{\mathrm{d}\theta'\mathrm{d}\theta''}{(\theta'\theta''(\theta'-\theta''))^{\beta}} \quad \left(\frac{\kappa_c}{\kappa_s}\right)^{2(3-\gamma)} \Phi\left[-(\theta'-\theta'')^{-\beta}\left(\frac{\kappa_c}{\kappa_s}\right)^2, 2, 3-\gamma\right] \\
&+ \left(\frac{\kappa}{\kappa_s}\right)^2 \iint_{\mathcal{D}_l} \frac{\mathrm{d}\theta'\mathrm{d}\theta''}{(\theta'\theta''(\theta'-\theta''))^{\beta}} \quad \left(\frac{\kappa_0}{\kappa_s}\right)^{2(3-\gamma)} \Phi\left[-(\theta'-\theta'')^{-\beta}\left(\frac{\kappa_0}{\kappa_s}\right)^2, 2, 3-\gamma\right] \\
&- 2\left(\frac{\kappa}{\kappa_s}\right)^2 \iint_{\mathcal{D}_l} \frac{\mathrm{d}\theta'\mathrm{d}\theta''}{(\theta'\theta''(\theta'-\theta''))^{\beta}} \left(\frac{\kappa_0\kappa_c}{\kappa_s^2}\right)^{3-\gamma} \Phi\left[-(\theta'-\theta'')^{-\beta}\frac{\kappa_0\kappa_c}{\kappa_s^2}, 2, 3-\gamma\right].
\end{aligned} \tag{S63}$$

One sees that these three terms are similar, and so we treat the general integral

$$I_\zeta = \iint_{\mathcal{D}_l} \mathrm{d}\theta'\mathrm{d}\theta'' \frac{\Phi\left[-(\theta'-\theta'')^{-\beta}\zeta, 2, 3-\gamma\right]}{(\theta'\theta''(\theta'-\theta''))^{\beta}} \zeta^{3-\gamma} = \iint_{\mathcal{D}_l} \mathrm{d}\theta'\mathrm{d}\theta'' \frac{\Phi\left[-\theta''^{-\beta}\zeta, 2, 3-\gamma\right]}{(\theta'\theta''(\theta'-\theta''))^{\beta}} \zeta^{3-\gamma}, \tag{S64}$$

where the transformation $\theta''' = \theta' - \theta''$, $\theta''' \to \theta''$ was performed. Now, the argument of the Lerch zeta function can in principle be smaller and larger than one. If it is smaller, it can be shown that $\Phi[-(\theta' - \theta'')^{-\beta}\zeta, 2, 3-\gamma] < 2^{\gamma-3}$. If it is bigger than one can use the identity described in Ref. [3]

$$\Phi[-z^2, 2, 3-\gamma] = z^{-2(3-\gamma)}\Big(2\psi(\gamma)\ln z + \vartheta(\gamma)\Big) + \frac{1}{z^2}\Phi\left[\frac{1}{z^2}, 2, \gamma-2\right], \tag{S65}$$

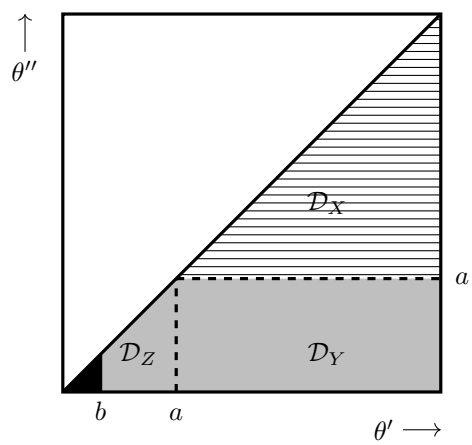
where

$$\psi(\gamma) = \Phi[-1, 1, 3-\gamma] + \Phi[-1, 1, \gamma-2] \quad \text{and} \quad \vartheta(\gamma) = -\pi^2\cot(\pi\gamma)\csc(\pi\gamma). \tag{S66}$$

The argument of the Lerch zeta function is exactly one, which is the inflection point between the behaviours, when

$$a = \zeta^{1/\beta} \tag{S67}$$

We must thus split the integration domain $\mathcal{D}_l$ in three regions (where $b = (\kappa_0/\kappa_s)^{2/\beta}$): $\mathcal{D}_X = [a, \pi] \times [a, \theta']$, $\mathcal{D}_Y = [a, \pi] \times [0, a]$ and $\mathcal{D}_Z = [b, a] \times [0, \theta']$ as depicted in Supplementary Figure 1. Now, the grey region is the one where the



Supplementary Figure 1: Integration regions. In the grey region ($\mathcal{D}_Y + \mathcal{D}_Z$) the argument of the Lerch zeta function is bigger than one, in the hatched region ($\mathcal{D}_X$) it is not and the black region is $\mathcal{D}_s$.

Lerch zeta function argument is bigger than one, in the hatched region we can bound the Lerch zeta function away and the black region is $\mathcal{D}_s$ and we thus do not care about it for the moment. Before going any further, let us note that Supplementary Figure 1 looks slightly different for different κ_c and ζ. If $\kappa_c \gg \kappa_s$ and $\zeta = (\kappa_c/\kappa_s)^2$, then $\zeta \gg 1$ and thus so is a. However, a as an integration limit must be smaller than π and thus in this case the $\mathcal{D}_X$ and $\mathcal{D}_Y$ regions disappear. When $\kappa_c \leq \kappa_s$ this is not the case as for all ζ, $a < \pi$. Finally, irrespective of the value of κ_c, for $\zeta = (\kappa_0/\kappa_s)^2$, $a = b$ and thus region $\mathcal{D}_Z$ vanishes.

Implementing the transformation given by Eq. (S65) in the grey region one obtains

$$I_\zeta \leq \iint \mathrm{d}\theta' \mathrm{d}\theta'' (\theta'(\theta'-\theta''))^{-\beta} \left\{ \theta''^{\beta(2-\gamma)} \left[\psi(\gamma) \ln\left(\frac{\zeta}{\theta''^{\beta}}\right) + \vartheta(\gamma) \right] + \zeta^{2-\gamma}(\gamma-2)^{-2} \right\}. \tag{S68}$$

As this leads to three different angular integrals, in the end we have seven different integrals to solve.

$$\begin{aligned}
\iint_{\mathcal{D}_X} \mathrm{d}\theta' \mathrm{d}\theta'' \left(\frac{1}{\theta'\theta''(\theta'-\theta'')}\right)^{\beta} &= \frac{a^{2-3\beta}}{3\beta-2} \Big\{ B_1[2\beta-1, 1-\beta] - B_1[1-\beta, 1-\beta] \\
&+ B_{\frac{a}{\pi}}[2\beta-1, 1-\beta] + (a/\pi)^{3\beta-2} B_{\frac{a}{\pi}}[1-\beta, 1-\beta] \Big\} \\
&+ \frac{4^{\beta-1/2}\pi^{5/2-3\beta}\Gamma[1-\beta]}{\Gamma[3/2-\beta](3\beta-2)} \left((a/\pi)^{2-3\beta} - 1\right) && \text{(S69)} \\
&= \boxed{c_{X_{11}} a^{2-3\beta} + c_{X_{12}}} && \text{(S70)}
\end{aligned}$$

$$\begin{aligned}
\iint_{\mathcal{D}_Y} \mathrm{d}\theta' \mathrm{d}\theta'' \left(\frac{1}{\theta'(\theta'-\theta'')}\right)^{\beta} &= \frac{a^{2-2\beta}}{2(\beta-1)^2} \Big\{ 2(\beta-1) B_{\frac{a}{\pi}}[2\beta-1, 1-\beta] \\
&- \pi^{-1/2}(\beta-1)\Gamma[1-\beta]\Gamma[\beta-1/2] - 1 \\
&+ (1 - {}_2F_1\left[\begin{matrix} 2(\beta-1), \beta \\ 2\beta-1 \end{matrix}; a/\beta\right])(a/\pi)^{2\beta-2} \Big\} && \text{(S71)} \\
&\simeq \boxed{c_{Y_{11}} a^{2-2\beta} + c_{Y_{12}}} && \text{(S72)}
\end{aligned}$$

$$\begin{aligned}
\iint_{\mathcal{D}_Y} \mathrm{d}\theta' \mathrm{d}\theta'' \left(\frac{\theta''^{2-\gamma}}{\theta'(\theta'-\theta'')}\right)^{\beta} &= \frac{a^{2-\gamma\beta}}{\gamma\beta-2} \Big\{ B_1[1+2\beta-\gamma\beta, 1-\beta] \\
&- B_1[2\beta-1, 1-\beta] + B_{\frac{a}{\pi}}[2\beta-1, 1-\beta] \\
&- (a/\pi)^{\gamma\beta-2} B_{\frac{a}{\pi}}[1+2\beta-\gamma\beta, 1-\beta] \Big\} && \text{(S73)} \\
&\simeq \boxed{c_{Y_{21}} a^{2-\gamma\beta} + c_{Y_{22}} a^{1+2\beta-\gamma\beta}} && \text{(S74)}
\end{aligned}$$

$$\begin{aligned}
\iint_{\mathcal{D}_Y} \mathrm{d}\theta' \mathrm{d}\theta'' \left(\frac{\theta''^{2-\gamma}}{\theta'(\theta'-\theta'')}\right)^{\beta} \ln\left(\frac{\zeta}{\theta''^{\beta}}\right) &= \frac{\beta a^{2-\beta\gamma}\pi^{1-2\beta}}{(\beta(\gamma-2)-1)(\beta\gamma-2)^2} \\
&\times \Big\{ \frac{4^{\beta-1}}{\pi^{\frac{3}{2}-2\beta}} (\beta(\gamma-2)-1)\Gamma[1-\beta]\Gamma\left[\beta - \frac{1}{2}\right] \\
&+ \frac{\pi^{2\beta-1}\Gamma[1-\beta]\Gamma[-\gamma\beta+2\beta+2]}{\Gamma[-\gamma\beta+\beta+2]}
\end{aligned}$$

$$\times \left(\frac{1+(\gamma\beta-2)(H_{\beta(2-\gamma)}-H_{1+\beta-\gamma\beta})}{\beta(\gamma-2)-1} \right)$$

$$- a^{2\beta-1} \left(\frac{(\beta(\gamma-2)-1)}{2\beta-1} {}_2F_1 \left[\begin{matrix} \beta, 2\beta-1 \\ 2\beta \end{matrix} ; \frac{a}{\pi} \right] \right.$$

$$\frac{(\beta\gamma-2)}{\beta(\gamma-2)-1} {}_3F_2 \left[\begin{matrix} \beta, -\gamma\beta+2\beta+1, -\gamma\beta+2 \\ \beta+1-\gamma\beta+2\beta+2, -\gamma\beta+2\beta+2 \end{matrix} ; \frac{a}{\pi} \right]$$

$$\left. + {}_2F_1 \left[\begin{matrix} \beta, \beta(-\gamma)+2\beta+1 \\ \beta(-\gamma)+2\beta+2 \end{matrix} ; \frac{a}{\pi} \right] \right) \Bigg\} \tag{S75}$$

$$\simeq \boxed{c_{Y_{31}} a^{2-\gamma\beta} + c_{Y_{32}} a^{1+2\beta-\gamma\beta}} \tag{S76}$$

$$\iint_{\mathcal{D}_Z} \mathrm{d}\theta' \mathrm{d}\theta'' \left(\frac{1}{\theta'(\theta'-\theta'')} \right)^{\beta} = \frac{a^{2-2\beta}-b^{2-2\beta}}{2(\beta-1)^2} = \boxed{c_{Z_{11}} a^{2-2\beta} + c_{Z_{12}} b^{2-2\beta}} \tag{S77}$$

$$\iint_{\mathcal{D}_Z} \mathrm{d}\theta' \mathrm{d}\theta'' \left(\frac{\theta''^{2-\gamma}}{\theta'(\theta'-\theta'')} \right)^{\beta} = \frac{\Gamma[1-\beta]\Gamma[-\gamma\beta+2\beta+1]\left(b^{2-\beta\gamma}-a^{2-\beta\gamma}\right)}{(\beta\gamma-2)\Gamma[-\gamma\beta+\beta+2]} \tag{S78}$$

$$= \boxed{c_{Z_{21}} a^{2-\gamma\beta} + c_{Z_{22}} b^{2-\gamma\beta}} \tag{S79}$$

$$\iint_{\mathcal{D}_Z} \mathrm{d}\theta' \mathrm{d}\theta'' \left(\frac{\theta''^{2-\gamma}}{\theta'(\theta'-\theta'')} \right)^{\beta} \ln\left(\frac{\zeta}{\theta''^{\beta}}\right) = \frac{\Gamma[1+2\beta-\gamma\beta]\Gamma[1-\beta]\beta}{(\beta\gamma-2)\Gamma[2+\beta-\gamma\beta]} \left(a^{2-\gamma\beta}-b^{2-\gamma\beta}\right)$$

$$\times \left\{ H_{\beta(2-\gamma)} - H_{1+\beta-\gamma\beta} + \frac{1}{\gamma\beta-2} - \frac{1}{\beta}\log(\zeta) \right.$$

$$\left. + \frac{a^{2-\gamma\beta}\log a - b^{2-\gamma\beta}\log b}{a^{2-\gamma\beta}-b^{2-\gamma\beta}} \right\} \tag{S80}$$

$$= \boxed{\begin{matrix} a^{2-\gamma\beta}\Big(c_{Z_{31}} + c_{Z_{32}}\left(\log(a) - \frac{1}{\beta}\log(\zeta)\right)\Big) \\ -b^{2-\gamma\beta}\Big(c_{Z_{31}} + c_{Z_{32}}\left(\log(b) - \frac{1}{\beta}\log(\zeta)\right)\Big) \end{matrix}} \tag{S81}$$

The next step is to organise the different scalings (see Supplementary Table (1), where we have defined $c_{Y_i} = c_{Y_{1i}} + c_{Y_{2i}} + c_{Y_{3i}}$ and similarly for Z) that were found and find which is dominant.

Let us note that as, the final results (Eq. (S63)) contains $I_{\kappa_c^2/\kappa_s^2} - 2I_{\kappa_0\kappa_c/\kappa_s^2}$, the terms containing $\ln(\kappa_c/\kappa_0)$ cancel. We now have many different scaling behaviours, and the question of which one dominates again depends on the value of β as well as κ_c. As a matter of fact, if one includes the κ_s^{-2} pre-factor in Eq. (S63), one recovers the same behaviour as was found for the lower bound

$$I_- \leq \begin{cases} C_{-,1}N^{-2/\beta+\gamma-1} + C_{-,2}N^{-1}\ln N & \text{if} \quad \kappa_c \gg \kappa_s \\ N^{-1}\Big(C_{-,3}N^{\gamma-2/\beta} + C_{-,4}N^{(1-\alpha)(\gamma-2/\beta)} + C_{-,5}N^{(1-\alpha)(\gamma-3)}\Big) & \text{if} \quad \kappa_c \leq \kappa_s \end{cases}, \tag{S82}$$

	$\zeta = \left(\frac{\kappa_0}{\kappa_s}\right)^2$	$\zeta = \left(\frac{\kappa_0\kappa_c}{\kappa_s^2}\right)$	$\zeta = \left(\frac{\kappa_c}{\kappa_s}\right)^2 \quad (\kappa_c \gg \kappa_s)$	$\zeta = \left(\frac{\kappa_c}{\kappa_s}\right)^2 \quad (\kappa_c \leq \kappa_s)$
X	$c_{X_{11}}\left(\frac{\kappa_0}{\kappa_s}\right)^{\frac{4}{\beta}-2\gamma} + c_{X_{12}}\left(\frac{\kappa_0}{\kappa_s}\right)^{2(3-\gamma)}$	$c_{X_{11}}\left(\frac{\kappa_0\kappa_c}{\kappa_s^2}\right)^{\frac{2}{\beta}-\gamma} + c_{X_{12}}\left(\frac{\kappa_0\kappa_c}{\kappa_s^2}\right)^{3-\gamma}$		$c_{X_{11}}\left(\frac{\kappa_c}{\kappa_s}\right)^{\frac{4}{\beta}-2\gamma} + c_{X_{12}}\left(\frac{\kappa_c}{\kappa_s}\right)^{2(3-\gamma)}$
Y	$c_{Y1}\left(\frac{\kappa_0}{\kappa_s}\right)^{\frac{4}{\beta}-2\gamma} + c_{Y2}\left(\frac{\kappa_0}{\kappa_s}\right)^{4+\frac{2}{\beta}-2\gamma}$	$c_{Y1}\left(\frac{\kappa_0\kappa_c}{\kappa_s^2}\right)^{\frac{2}{\beta}-\gamma} + c_{Y2}\left(\frac{\kappa_0\kappa_c}{\kappa_s^2}\right)^{2+1/\beta-\gamma}$		$c_{Y1}\left(\frac{\kappa_c}{\kappa_s}\right)^{\frac{4}{\beta}-2\gamma} + c_{Y2}\left(\frac{\kappa_c}{\kappa_s}\right)^{4+2/\beta-2\gamma}$
Z		$c_{Z1}\left(\frac{\kappa_0\kappa_c}{\kappa_s^2}\right)^{\frac{2}{\beta}-\gamma} + (c_{Z_{22}} - c_{Z_{31}})\left(\frac{\kappa_0}{\kappa_s}\right)^{\frac{4}{\beta}-2\gamma} + \frac{c_{Z_{32}}}{\beta}\left(\frac{\kappa_0}{\kappa_s}\right)^{\frac{4}{\beta}-2\gamma}\ln\left(\frac{\kappa_c}{\kappa_0}\right) + c_{Z_{12}}\left(\frac{\kappa_0\kappa_c}{\kappa_s^2}\right)^{2-\gamma}\left(\frac{\kappa_0}{\kappa_s}\right)^{\frac{4}{\beta}-4}$	$c_{Z_{11}}\pi^{2-2\beta}\left(\frac{\kappa_c}{\kappa_s}\right)^{2(2-\gamma)} + (c_{Z_{21}} + c_{Z_{31}})\pi^{2-\gamma\beta} + (c_{Z_{22}} - c_{Z_{32}})\left(\frac{\kappa_0}{\kappa_s}\right)^{\frac{4}{\beta}-2\gamma} - \frac{2}{\beta}c_{Z_{32}}\pi^{2-\gamma\beta}\ln\left(\frac{\kappa_c}{\kappa_s}\right) + \frac{2}{\beta}c_{Z_{32}}\left(\frac{\kappa_0}{\kappa_s}\right)^{\frac{4}{\beta}-2\gamma}\ln\left(\frac{\kappa_c}{\kappa_0}\right) + c_{Z_{23}}\pi^{2-\gamma\beta}\ln(\pi) + c_{Z_{12}}\left(\frac{\kappa_c}{\kappa_s}\right)^{2(2-\gamma)}\left(\frac{\kappa_0}{\kappa_s}\right)^{\frac{4}{\beta}-4}$	$c_{Z1}\left(\frac{\kappa_c}{\kappa_s}\right)^{\frac{4}{\beta}-2\gamma} + (c_{Z_{22}} - c_{Z_{31}})\left(\frac{\kappa_0}{\kappa_s}\right)^{\frac{4}{\beta}-2\gamma} + \frac{2c_{Z_{32}}}{\beta}\left(\frac{\kappa_0}{\kappa_s}\right)^{\frac{4}{\beta}-2\gamma}\ln\left(\frac{\kappa_c}{\kappa_0}\right) + c_{Z_{12}}\left(\frac{\kappa_c}{\kappa_s}\right)^{4-2\gamma}\left(\frac{\kappa_0}{\kappa_s}\right)^{\frac{4}{\beta}-4}$

Supplementary Table 1: The different terms resulting from (S63).

where $C_{-,i}$ are constants.

This seems to go in the right direction. However, we have not explored the full integration domain yet. It turns out though that the integration domain $\mathcal{D}_s$ does not lead to any new scaling:

$$\begin{aligned}
I_{-,s} &= \iint \mathrm{d}x\mathrm{d}y(xy)^{-\gamma}\iint_{\mathcal{D}_s}\frac{1}{1+\frac{\theta'^{\beta}\kappa_s}{\kappa x}}\frac{1}{1+\frac{\theta''^{\beta}\kappa_s}{\kappa y}}\frac{1}{1+\frac{(\theta'-\theta'')^{\beta}}{xy}} \\
&\leq \iint \mathrm{d}x\mathrm{d}y(xy)^{-\gamma}\iint_{\mathcal{D}_s}1 \\
&= \left(\frac{\kappa_0}{\kappa_s}\right)^{4/\beta}\iint \mathrm{d}x\mathrm{d}y(xy)^{-\gamma} \\
&= \left(\frac{\kappa_0}{\kappa_s}\right)^{4/\beta}\frac{1}{(1-\gamma)^2}\left(\left(\frac{\kappa_c}{\kappa_s}\right)^{1-\gamma} - \left(\frac{\kappa_0}{\kappa_s}\right)^{1-\gamma}\right)^2 \\
&\simeq \frac{1}{(1-\gamma)^2}\left(\frac{\kappa_0}{\kappa_s}\right)^{2(1-\gamma+2/\beta)} \\
&\sim N^{-1+\gamma-2/\beta}.
\end{aligned} \tag{S83}$$

The contribution of $\mathcal{D}_s$ is thus subleading for small β and equally dominant as the other contributions for large β. We have thus shown that for the the upper and lower bound the dominant scaling is the same. We now have the scaling of all distinct

parts, $B, A_+, A-$, so we can now combine them all.

$$\boxed{\overline{c} \simeq \begin{cases} C_1 N^{2-2/\beta} + C_2 N^{2-\gamma} \ln N & \text{if} \quad \kappa_c \gg \kappa_s \\ C_3 N^{2-2/\beta} + C_4 N^{2-2/\beta-\alpha(\gamma-2/\beta)} + C_5 N^{-1-\alpha(\gamma-3)} & \text{if} \quad \kappa_c \leq \kappa_s \end{cases}} \tag{S84}$$

Let us discuss the limiting cases of α. When $\alpha = 0$, $\kappa_0 \sim \kappa_c$, and the network thus has a homogeneous degree distribution. Then, $C \simeq (C_3 + C_4)N^{2-2/\beta} + C_5 N^{-1}$. If $\alpha = 1$, i.e. $\kappa_c \sim \kappa_s$, the scaling becomes $C \simeq C_3 N^{2-2/\beta} + (C_4 + C_5)N^{2-\gamma}$.

Supplementary Note 1.3.3 Case $\beta = 1$

We now turn to the limit $\beta = 1$. The general practice of finding upper and lower bounds for the various relevant integrals will be again pursued here, and in many cases the integrals examined will be similar to the ones studied above. However, there are some important differences that force us to treat this case separately. For one, we know that in the case of $\beta = 1$, μ scales as $\hat{\mu} \sim (\ln N)^{-1}$ instead of $\hat{\mu} \sim N^{1-\beta}$, and thus $\kappa_s \sim \sqrt{N \ln N}$, which of course alters scaling. We will represent all integrals evaluated at $\beta = 1$ by a tilde ($\tilde{A}_-, \tilde{A}_+, \tilde{B}$). We start with $\tilde{B}$:

$$\begin{aligned} \tilde{B} &= \pi \left(\frac{\kappa_c}{\kappa_s}\right)^{1-\gamma} \int_0^1 \mathrm{d}\theta \int_{\kappa_0/\kappa_c}^1 \mathrm{d}x \frac{x^{-\gamma}}{1 + \dfrac{\pi\theta\kappa_s^2}{\kappa_c \kappa x}} \\ &= \pi \left(\frac{\kappa_c}{\kappa_s}\right)^{1-\gamma} \left\{ \frac{\kappa\kappa_c}{\pi\kappa_s^2} \frac{\log\left(1 + \frac{\pi\kappa_s^2}{\kappa_c\kappa}\right)}{2-\gamma} + \frac{1}{\gamma-2}\left(\frac{\kappa_0}{\kappa_c}\right)^{2-\gamma} \frac{\kappa\kappa_c}{\pi\kappa_s^2} \log\left(1 + \frac{\pi\kappa_s^2}{\kappa\kappa_0}\right) \right. \\ &\left. + \frac{1}{(\gamma-2)(\gamma-1)} \left({}_2F_1\left[\begin{matrix} 1, \gamma-1 \\ \gamma \end{matrix} ; -\frac{\pi\kappa_s^2}{\kappa\kappa_c}\right] - \left(\frac{\kappa_0}{\kappa_c}\right)^{1-\gamma} {}_2F_1\left[\begin{matrix} 1, \gamma-1 \\ \gamma \end{matrix} ; -\frac{\pi\kappa_s^2}{\kappa\kappa_0}\right] \right) \right\}. \end{aligned} \tag{S85}$$

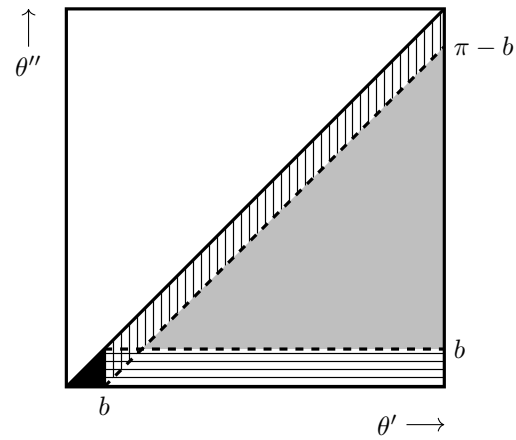
The second term is dominant and thus $\tilde{B}$ scales as

$$\tilde{B} \sim \kappa_s^{\gamma-3} \log(\kappa_s) \sim N^{\frac{\gamma-3}{2}} (\log N)^{\frac{\gamma-1}{2}}. \tag{S86}$$

For the lower bound of the numerator of the clustering coefficient we can use the result found in Eq. (S62) as nowhere was it assumed that $\beta < 1$. Irrespective of κ_c this gives us

$$\tilde{A}_+ \leq \tilde{c}_+ N^{\gamma-3} (\ln N)^{\gamma-3}. \tag{S87}$$

For the upper bound of $\tilde{A}_-$ we cannot follow the same path as was done in the case of general β. This is because the upper bound employed, given by Eq. (S63), diverges in the $\beta = 1$ limit. Thus, we must find a stricter bound. This is done by once again dividing the angular integration domain, this time in four pieces: $\mathcal{D}_s = [0, (\kappa_0/\kappa_s)^2] \times [0, \theta']$, $\mathcal{D}_2 = [(\kappa_0/\kappa_s)^2, \pi] \times [0, (\kappa_0/\kappa_s)^2]$, $\mathcal{D}_3 = [(\kappa_0/\kappa_s)^2, \pi] \times [\theta' - (\kappa_0/\kappa_s)^2, \theta']$ and $\mathcal{D}_3 = [2(\kappa_0/\kappa_s)^2, \pi] \times [(\kappa_0/\kappa_s)^2, \theta' - (\kappa_0/\kappa_s)^2]$,



Supplementary Figure 2: Integration regions where $b = \frac{\kappa_0^2}{\kappa_s^2}$ The black region is region $\mathcal{D}_s$. The horizontally striped region is region $\mathcal{D}_2$. The vertically striped region is region $\mathcal{D}_3$. The grey region is region $\mathcal{D}_4$.

as represented in Supplementary Figure 2. Note that regions $\mathcal{D}_2$ and $\mathcal{D}_3$ overlap, but that is not a problem as our integrand is positive and counting a region double just increases the value of the integral, which in turn work for our purposes as we are only looking for an upper bound. For the region $\mathcal{D}_s$ we can use the result (S83):

$$\tilde{A}_{-,s} \leq \tilde{c}_{-,s} N^{\gamma-3} \left(\ln N\right)^{\gamma-3}. \tag{S88}$$

Turning to $\mathcal{D}_2$ we obtain

$$\begin{aligned}\tilde{A}_{-,2} &= \iint \mathrm{d}x\mathrm{d}y (xy)^{-\gamma} \iint_{\mathcal{D}_2} \frac{\mathrm{d}\theta'\mathrm{d}\theta''}{1+\dfrac{\theta'\kappa_s}{\kappa x}} \frac{1}{1+\dfrac{\theta''\kappa_s}{\kappa y}} \frac{1}{1+\dfrac{\theta'-\theta''}{xy}} \\ &\leq \frac{\kappa}{\kappa_s} \iint_{\mathcal{D}_2} \frac{\mathrm{d}\theta'\mathrm{d}\theta''}{\theta'} \int\limits_{\kappa_s/\kappa_c}^{\kappa_s/\kappa_0} \mathrm{d}x \int\limits_{\kappa_s/\kappa_c}^{\kappa_s/\kappa_0} \mathrm{d}y \frac{x^{\gamma-3}y^{\gamma-2}}{1+xy(\theta'-\theta'')}\end{aligned} \tag{S89}$$

where we have bounded the integral by decreasing the size of the denominators of the first and second terms. We also performed a change of variables of x and y. We now extend the lower bounds of the x and y integrals to zero, which can be done as our integral is positive, and so the resulting integral will be larger or equal to the original one.

$$\begin{aligned}\tilde{A}_{-,2} &\leq \frac{\kappa}{\kappa_s} \iint_{\mathcal{D}_2} \frac{\mathrm{d}\theta'\mathrm{d}\theta''}{\theta'} \int\limits_{0}^{\kappa_s/\kappa_0} \mathrm{d}x \int\limits_{0}^{\kappa_s/\kappa_0} \mathrm{d}y \frac{x^{\gamma-3}y^{\gamma-2}}{1+xy(\theta'-\theta'')} \\ &= \frac{\kappa}{\kappa_s} (\kappa_s/\kappa_0)^{2\gamma-3} \iint_{\mathcal{D}_2} \frac{\mathrm{d}\theta'\mathrm{d}\theta''}{\theta'} \left(\Phi\left[-\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta''), 1, \gamma-2\right] - \Phi\left[-\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta''), 1, \gamma-1\right] \right)\end{aligned} \tag{S90}$$

We know again have the situation that depending on the values of the angular coordinates, the arguments of the Φ's diverge or go to zero. For the region $\mathcal{D}_{2s} = [b, 2b] \times [0, b]$, $\theta' - \theta'' \in [0, b]$, so the argument lies between zero and one. For the

region $\mathcal{D}_{2l} = [2b, \pi] \times [0, b]$, $\theta' - \theta'' \in [b, \pi]$, so the argument is larger than one. We first turn to the second region. Here the argument can diverge and we should thus perform a similar transformation as Eq. (S65). It is not exactly the same as the second argument of the Φ's is now 1 and not two 2, but the derivation is equivalent. This leads us to

$$\begin{aligned}
&\frac{\kappa}{\kappa_s}(\kappa_s/\kappa_0)^{2\gamma-3}\iint_{\mathcal{D}_{2l}}\frac{\mathrm{d}\theta'\mathrm{d}\theta''}{\theta'}\left(\Phi\left[-\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta''),1,\gamma-2\right]-\Phi\left[-\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta''),1,\gamma-1\right]\right)\\
&=\frac{\kappa}{\kappa_s}(\kappa_s/\kappa_0)^{2\gamma-3}\iint_{\mathcal{D}_{2l}}\frac{\mathrm{d}\theta'\mathrm{d}\theta''}{\theta'}\Bigg(\left(\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta'')\right)^{2-\gamma}\left(\Phi[-1,1,3-\gamma]+\Phi[-1,1,2-\gamma]\right)\\
&\qquad-\left(\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta'')\right)^{-1}\Phi\left[-\left(\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta'')\right)^{-1},1,3-\gamma\right]\\
&\qquad+\left(\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta'')\right)^{1-\gamma}\left(\Phi[-1,1,2-\gamma]+\Phi[-1,1,1-\gamma]\right)\\
&\qquad-\left(\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta'')\right)^{-1}\Phi\left[-\left(\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta'')\right)^{-1},1,2-\gamma\right]\Bigg)\\
&\leq\frac{\kappa}{\kappa_s}(\kappa_s/\kappa_0)^{2\gamma-3}\iint_{\mathcal{D}_{2l}}\frac{\mathrm{d}\theta'\mathrm{d}\theta''}{\theta'}\Bigg(\left(\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta'')\right)^{2-\gamma}\left(\Phi[-1,1,3-\gamma]+\Phi[-1,1,2-\gamma]\right)\\
&\qquad+\left(\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta'')\right)^{1-\gamma}\left(\Phi[-1,1,2-\gamma]+\Phi[-1,1,1-\gamma]\right)\\
&\qquad-2\left(\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta'')\right)^{-1}\Bigg)\sim\kappa_s^{2(\gamma-3)}\sim N^{\gamma-3}\left(\ln N\right)^{\gamma-3}.
\end{aligned}\tag{S91}$$

For $\mathcal{D}_{2s}$ we can immediately bound away the Φ to find

$$\begin{aligned}
&\frac{\kappa}{\kappa_s}(\kappa_s/\kappa_0)^{2\gamma-3}\iint_{\mathcal{D}_{2s}}\frac{\mathrm{d}\theta'\mathrm{d}\theta''}{\theta'}\left(\Phi\left[-\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta''),1,\gamma-2\right]-\Phi\left[-\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta''),1,\gamma-1\right]\right)\\
&\leq\frac{\kappa}{\kappa_s}(\kappa_s/\kappa_0)^{2\gamma-3}\iint_{\mathcal{D}_{2s}}\frac{\mathrm{d}\theta'\mathrm{d}\theta''}{\theta'}=\frac{\kappa}{\kappa_s}(\kappa_s/\kappa_0)^{2\gamma-5}\ln 2\sim N^{\gamma-3}\left(\ln N\right)^{\gamma-3}.
\end{aligned}\tag{S92}$$

Combining the two results we find that $\tilde{A}_{-,2}\leq\tilde{c}_{-,2}N^{\gamma-3}\left(\ln N\right)^{\gamma-3}$ as expected. Then we investigate to $\mathcal{D}_3$:

$$\begin{aligned}
\tilde{A}_{-,3}&=\iint\mathrm{d}x\mathrm{d}y(xy)^{-\gamma}\iint_{\mathcal{D}_3}\mathrm{d}\theta'\mathrm{d}\theta''\frac{1}{1+\dfrac{\theta'\kappa_s}{\kappa x}}\frac{1}{1+\dfrac{\theta''\kappa_s}{\kappa y}}\frac{1}{1+\dfrac{\theta'-\theta''}{xy}}\\
&=\iint\mathrm{d}x\mathrm{d}y(xy)^{-\gamma}\iint_{\mathcal{D}_2}\mathrm{d}\theta'\mathrm{d}\theta''\frac{1}{1+\dfrac{\theta'\kappa_s}{\kappa x}}\frac{1}{1+\dfrac{(\theta'-\theta'')\kappa_s}{\kappa y}}\frac{1}{1+\dfrac{\theta''}{xy}}\\
&\leq\left(\frac{\kappa}{\kappa_s}\right)^2\iint\mathrm{d}x\mathrm{d}yx^{1-\gamma}y^{1-\gamma}\iint_{\mathcal{D}_2}\mathrm{d}\theta'\mathrm{d}\theta''\frac{1}{\theta'}\frac{1}{\theta'-\theta''}\\
&=\frac{1}{(2-\gamma)^2}\left(\frac{\kappa}{\kappa_s}\right)^2\left(\frac{\kappa_0}{\kappa_s}\right)^{2(2-\gamma)}\left(\frac{\pi^2}{6}-\mathrm{Li}_2\left[\frac{\kappa_0^2}{\kappa_s^2\pi}\right]\right)\\
&\sim\kappa_s^{2(\gamma-3)}\sim N^{\gamma-3}\left(\ln N\right)^{\gamma-3}.
\end{aligned}\tag{S93}$$

Here $\mathrm{Li}_2(z)$ is the dilogarithm. The final region to be studied is $\mathcal{D}_4$:

$$\begin{aligned}
\tilde{A}_{-,4} &= \iint \mathrm{d}x\mathrm{d}y(xy)^{-\gamma} \iint_{\mathcal{D}_4} \mathrm{d}\theta'\mathrm{d}\theta'' \frac{1}{1+\frac{\theta'\kappa_s}{\kappa x}} \frac{1}{1+\frac{\theta''\kappa_s}{\kappa y}} \frac{1}{1+\frac{\theta'-\theta''}{xy}} \\
&\leq \left(\frac{\kappa}{\kappa_s}\right)^2 \iint \mathrm{d}x\mathrm{d}y(xy)^{1-\gamma} \iint_{\mathcal{D}_4} \mathrm{d}\theta'\mathrm{d}\theta'' \frac{1}{\theta'\theta''} \frac{1}{1+\frac{\theta'-\theta''}{xy}} \\
&= \left(\frac{\kappa}{\kappa_s}\right)^2 \int\limits_{\kappa_s/\kappa_c}^{\kappa_s/\kappa_0} \mathrm{d}x \int\limits_{\kappa_s/\kappa_c}^{\kappa_s/\kappa_0} \mathrm{d}y(xy)^{\gamma-3} \iint_{\mathcal{D}_4} \mathrm{d}\theta'\mathrm{d}\theta'' \frac{1}{\theta'\theta''} \frac{1}{1+xy(\theta'-\theta'')} \\
&\leq \left(\frac{\kappa}{\kappa_s}\right)^2 \int\limits_{0}^{\kappa_s/\kappa_0} \mathrm{d}x \int\limits_{0}^{\kappa_s/\kappa_0} \mathrm{d}y(xy)^{\gamma-3} \iint_{\mathcal{D}_4} \mathrm{d}\theta'\mathrm{d}\theta'' \frac{1}{\theta'\theta''} \frac{1}{1+xy(\theta'-\theta'')} \\
&= \left(\frac{\kappa}{\kappa_s}\right)^2 \left(\frac{\kappa_s}{\kappa_0}\right)^{2(\gamma-2)} \iint_{\mathcal{D}_4} \mathrm{d}\theta'\mathrm{d}\theta'' \frac{1}{\theta'\theta''} \Phi\left[-\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta''), 2, \gamma-2\right] \\
&\leq \left(\frac{\kappa}{\kappa_s}\right)^2 \iint_{\mathcal{D}_4} \mathrm{d}\theta'\mathrm{d}\theta'' \frac{1}{\theta'\theta''} \Bigg\{ (\theta'-\theta'')^{2-\gamma} \left[\Psi(.)\log\left(\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta'')\right) + \vartheta(.)\right] \\
&+ \left(\frac{\kappa_s}{\kappa_0}\right)^{2(\gamma-3)} (\theta'-\theta'')^{-1}(3-\gamma)^{-2} \Bigg\}. \qquad \text{(S94)}
\end{aligned}$$

Let us investigate the term with the logarithm first.

$$\begin{aligned}
&\left(\frac{\kappa}{\kappa_s}\right)^2 \int\limits_{2\kappa_0^2/\kappa_s^2}^{\pi} \mathrm{d}\theta' \int\limits_{\kappa_0^2/\kappa_s^2}^{\theta'-\kappa_0^2/\kappa_s^2} \mathrm{d}\theta'' \frac{(\theta'-\theta'')^{2-\gamma}}{\theta'\theta''} \log\left(\frac{\kappa_s^2}{\kappa_0^2}(\theta'-\theta'')\right) \\
&= \left(\frac{\kappa}{\kappa_s}\right)^2 \left(\frac{\kappa_0}{\kappa_s}\right)^{2(2-\gamma)} \int\limits_{2}^{\pi\kappa_s^2/\kappa_0^2} \mathrm{d}\theta' \int\limits_{1}^{\theta'-1} \mathrm{d}\theta'' \frac{(\theta'-\theta'')^{2-\gamma}}{\theta'\theta''} \log(\theta'-\theta'') \\
&= \left(\frac{\kappa}{\kappa_s}\right)^2 \left(\frac{\kappa_0}{\kappa_s}\right)^{2(2-\gamma)} \int\limits_{2}^{\pi\kappa_s^2/\kappa_0^2} \mathrm{d}\theta' \int\limits_{1}^{\theta'-1} \mathrm{d}\theta'' \frac{(\theta'')^{2-\gamma}}{\theta'(\theta'-\theta'')} \log(\theta''). \qquad \text{(S95)}
\end{aligned}$$

This can then be evaluated. The θ'' integral leads to a variety of different terms, which need to be treated separately. Some variable transformations need to be performed, and some special functions need to be expanded to their series representation. It can be shown that the integral to leading order is constant in N, implying that the logarithm term of $\tilde{A}_{-,4}$ scales as $\kappa_s^{2(\gamma-3)}$. The other two terms in expression (S94) are easier to evaluate:

$$\begin{aligned}
\iint_{\mathcal{D}_4} \mathrm{d}\theta'\mathrm{d}\theta'' \frac{1}{\theta'\theta''}(\theta'-\theta'')^{2-\gamma} &= \frac{\left(b^{2-\gamma}+\pi^{2-\gamma}\right)}{\gamma-2}\left\{B_{1-\frac{b}{\pi}}[3-\gamma,\gamma-2] - B_{\frac{1}{2}}[3-\gamma,\gamma-2]\right\} \\
&+ \frac{b^{2-\gamma}\ln\left(2-\frac{2b}{\pi}\right)}{\gamma-2} \sim \left(\frac{\kappa_0}{\kappa_s}\right)^{2(2-\gamma)} \qquad \text{(S96)}
\end{aligned}$$

$$\iint_{\mathcal{D}_4} \mathrm{d}\theta'\mathrm{d}\theta'' \frac{1}{\theta'\theta''}(\theta'-\theta'')^{-1} = \frac{2\log\left(2-\frac{2b}{\pi}\right)}{b} - \frac{2\log\left(\frac{\pi}{b}-1\right)}{\pi} \sim \frac{\kappa_s^2}{\kappa_0^2}. \qquad \text{(S97)}$$

Plugging this back in we find that also the integral over the region $\mathcal{D}_4$ scales as $N^{\gamma-3}(\ln N)^{\gamma-3}$.

Thus, we can finally conclude that for $\beta = 1$, the clustering coefficient must scale as

$$\boxed{\overline{c} \sim \frac{N^{\gamma-3}(\log N)^{\gamma-3}}{N^{\gamma-3}(\log N)^{\gamma-1}} = (\log N)^{-2}}. \tag{S98}$$

With this we have found the critical exponent $\eta/\nu = 2$.

Supplementary Note 1.4 Exponent η

In this section we show that the scaling exponent η that encodes how the clustering approaches zero when $\beta \to \beta_c^+ = 1$. As this only requires working on the low temperature side of the transition, we can directly work in the thermodynamic limit (we thus take first the limit $N \to \infty$ and then $\beta \to 1$). To this end, we denote the general definition of the clustering coefficient with hidden degree κ and (without loss of generality) spacial coordinate $r = 0$

$$\overline{c}(\kappa) = \frac{\int_{\kappa_0}^{\infty} \mathrm{d}\kappa' \int_{\kappa_0}^{\infty} \mathrm{d}\kappa'' \int_{-\infty}^{\infty} \mathrm{d}r' \int_{-\infty}^{\infty} \mathrm{d}r'' \rho(\kappa')\rho(\kappa'')p(\kappa,\kappa',|r'|)p(\kappa,\kappa'',|r''|)p(\kappa',\kappa'',|r'-r''|)}{\left(\int_{\kappa_0}^{\infty} \mathrm{d}\kappa' \int_{-\infty}^{\infty} \mathrm{d}r' \rho(\kappa')p(\kappa,\kappa',|r'|)\right)^2}. \tag{S99}$$

where we can use connection probability (S5) and $\hat{\mu}$ (S10).

Let us first turn to the denominator:

$$\int \mathrm{d}\kappa' \rho(\kappa') \int_{-\infty}^{\infty} \frac{\mathrm{d}r'}{1 + \left(\frac{r'}{\kappa\kappa'\hat{\mu}}\right)^{\beta}} = \kappa, \tag{S100}$$

where we have plugged in the definition of $\hat{\mu}$ and used that $\langle k \rangle = \frac{\gamma-1}{\gamma-2}\kappa_0$.

The next step is the numerator. We first perform the transformation $t = r'/(\kappa\kappa'\hat{\mu})$ and $\tau = r''/(\kappa\kappa''\hat{\mu})$ to obtain

$$\overline{c}(\kappa) = \frac{\hat{\mu}^2}{4}(\gamma-1)^2\kappa_0^{2\gamma-2} \iiiint \mathrm{d}\kappa'\mathrm{d}\kappa''\mathrm{d}t\mathrm{d}\tau \frac{(\kappa'\kappa'')^{1-\gamma}}{1+|t|^{\beta}} \frac{1}{1+|\tau|^{\beta}} \frac{1}{1+\left|\frac{\kappa t}{\kappa''} - \frac{\kappa\tau}{\kappa'}\right|^{\beta}}. \tag{S101}$$

We know that $\hat{\mu}^2 \sim (\beta-1)^2$. This is exactly the scaling that we expect from numerical investigation for the clustering coefficient. Thus, all we need to prove is that at $\beta = 1$, the numerator is finite. If so, its $(\beta - 1)$ dependence must be order $\mathcal{O}(1)$. If the full expression contained $(\beta-1)^{-n}$ terms with $n > 0$ it would diverge at the critical point and if the dominant term was $\mathcal{O}((\beta-1)^n)$ with $n > 0$ the numerator would go to zero at the critical point. And indeed, numerical integration shows that at $\beta = 1$ the numerator is finite, leading to the conclusion that

$$\boxed{\overline{c}(\kappa) \sim (\beta-1)^2} \tag{S102}$$

such that $\eta = 2$, which in turn implies that $\nu = 1$.

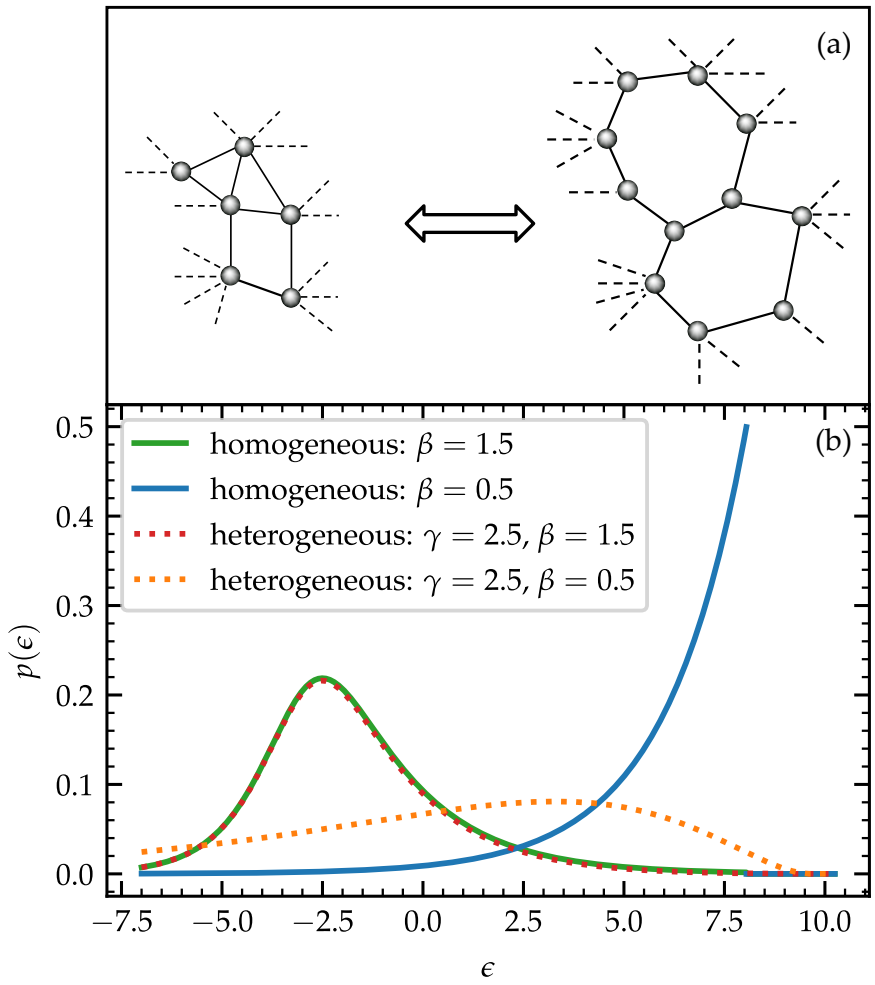
Supplementary Note 2 Real Networks

As was stated in the main text, the DPG algorithm can be used to find the temperature of its embedding in the $\mathbb{S}_1$ model. We list in Supplementary Table 2 a collection of real networks and their corresponding inverse temperatures. We choose to restrict ourselves to models where the inverse temperature lies below or close to the transition point β_c. We also show in Supplementary Figure 4 that important network measures are respected by the embedding.

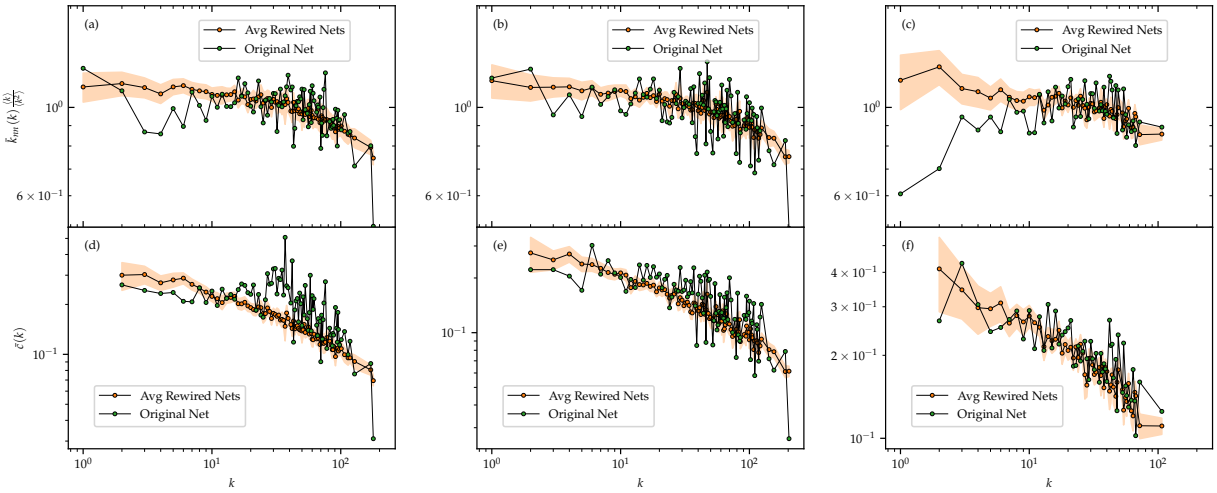
Network Names	Type	$\|V\|$	$\|E\|$	$\langle k \rangle$	Target $\overline{c}$	β
CElegans-C [7]	Biological - Brain	279	2287	16	0.34	1.5
Drosophila1-C [8]	Biological - Brain	350	2887	16	0.25	1.1
Drosophila2-C [8]	Biological - Brain	1770	8905	10	0.33	1.1
Arabidopsis-G [9]	Biological - Cell	4519	10721	4.7	0.16	1.2
CElegans-G [7]	Biological - Cell	3692	7650	4.2	0.11	0.77
Drosophila-G [10]	Biological - Cell	8114	38909	9.6	0.12	1.1
Human1-P [11]	Biological - Cell	913	7472	16	0.23	1.0
Human2-P [11]	Biological - Cell	1090	9369	17	0.20	1.0
Mus-G [10]	Biological - Cell	7402	16858	4.6	0.13	1.1
Rattus-G [10]	Biological - Cell	2350	3484	3.0	0.22	0.74
Yeast1-P [12]	Biological - Cell	1647	2518	3.1	0.10	1.2
Yeast2-P [13]	Biological - Cell	1458	1948	2.7	0.14	1.5
Polblogs-H [14]	Citation - Hyperlinks	1222	16714	27	0.36	1.1
Wiki-H [15]	Citation - Hyperlinks	1872	15367	16	0.42	1.3
Ecological [16]	Ecological - Troffic	700	6495	18	0.10	0.15
Commodities [17]	Economic - Commodities	374	1090	5.8	0.22	1.2
Friends-OFF [18]	Social Offline - Friends	2539	10455	8.2	0.15	1.4
Airports1 [19]	Transport - Flights	1572	17214	22	0.64	1.4

Supplementary Table 2: Properties of a selection of networks with the inverse temperature β obtained with the DPG algorithm. Only networks with $\beta < 1.5$ are shown.

Supplementary Note 3 Figures



Supplementary Figure 3: Panel (b) shows the probability $p(\epsilon)$ of finding a link with energy ϵ based on Eq. (S33). The full lines show the homogeneous case whereas the dotted lines represent the heterogeneous case with $\gamma = 2.5$. For both degree distributions we plot the $p(\epsilon)$ for both $\beta = 0.5$ (blue/orange) and $\beta = 1.5$ (green/red). In all cases $N = 10^5$ and $\langle k \rangle = 4$, i.e. this represents the situation for a sparse graph. Panel (a) illustrates the consequences of this transition for the cycles in the network, going from short cycles for $\beta > 1$ to long cycles for $\beta < 1$.



Supplementary Figure 4: Degree-degree correlations (panels (a), (b) and (c)) and average clustering coefficient per degree (panels (d), (e) and (f)) for three of the real networks in Supplementary Table 2. Panels (a) and (d) correspond to the Human1-P network, (b) and (e) to Human2-P and (c) and (f) to Drosophila-G. The green points represent the network measures corresponding to the original network. The orange points represent the the average of 100 randomized networks at the β that reproduces the correct global clustering coefficient (see Supplementary Table 2).

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