Generalization of the persistent random walk to dimensions greater than 1

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We propose a generalization of the persistent random walk for dimensions greater than 1. Based on a cubic lattice, the model is suitable for an arbitrary dimension *d*. We study the continuum limit and obtain the equation satisfied by the probability density function for the position of the random walker. An exact solution is obtained for the projected motion along an axis. This solution, which is written in terms of the free-space solution of the one-dimensional telegrapher's equation, may open a new way to address the problem of light propagation through thin slabs. [S1063-651X(98)00312-2]

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I. INTRODUCTION

The persistent random walk (PRW), first introduced by Fürth [1] and shortly after by Taylor [2], is probably the earliest and simplest generalization of the ordinary random walk that incorporates some form of momentum in addition to random motion. The persistent random walk differs from the ordinary random walk in that the probabilistic quantity used at each step is the probability of continually moving in a given direction rather than the probability of moving in a given direction regardless of the direction of the preceding step. In this way the PRW introduces some form of momentum, i.e., persistence, into the purely random motion. This remarkable feature of the model is one of the reasons why the PRW has been recently applied to describe scattering and diffusion in disordered media [3].

For one-dimensional lattices and in the continuum or diffusive limit the probability density function for the displacement at time t, $p(x,t)$, satisfies the telegrapher's equation (TE) [4]:

$$\frac{\partial^2 p}{\partial t^2}+2\lambda\frac{\partial p}{\partial t}=v^2\frac{\partial^2 p}{\partial x^2}. \tag{1}$$

As is well known, Eq. (1) has solutions with a finite velocity of propagation given by v [6]. This fact has justified the extensive use of TE as a generalization of the mesoscopic diffusion equations in fields such as heat propagation [5] and light dispersion in turbid media [6,7]. However, none of the generalizations explored in two and three dimensions of persistent random walks obeys the TE in the continuum limit [8–11].

On the other hand, and besides the recent work of Godoy *et al.* [10] on two-dimensional walks, there have been, to our knowledge, very few attempts to generalize the PRW to dimensions higher than 1 in spite of its potential for modeling transport in disordered media. One of the reasons for the lack of such a generalization is the nonexistence of a unique generalization of the PRW to dimensions greater than 1 since several kinds of lattices (cubic, hexagonal, etc.) can be used for the extension. Our main goal in this paper is to propose a generalization of the PRW to higher dimensions assuming a cubic lattice, and to obtain the governing equations for the probability density function of the process in the continuum limit.

Unfortunately, in the continuum limit and for dimensions greater than 1, the probability density function of the process does not obey a higher-dimensional telegrapher's equation. Nevertheless, in the context of transport in disordered media, the partial differential equation describing particle concentration (i.e., the probability density function) can suggest new approximations for the transport equation of more realistic models. This is the case, for example, of light propagation in turbid media where such an approach becomes extremely useful, especially when photons propagate in constrained geometries such as thin slabs where the Gaussian approximation becomes quite imprecise [12]. In addition, the telegrapher's equation has been shown not to furnish better results than the diffusion approximation in two and three dimensions [13].

As we have mentioned, in the one-dimensional case the equation satisfied by $p(x,t)$ is the telegrapher's equation. In two dimensions, the model considered herein was partially analyzed by Godoy *et al.* [10] but the equation for the probability density function, $p(\boldsymbol{r},t)$, of the process was not obtained. Another goal of this paper is to study the projected motion, along a given direction, of the higher-dimensional PRW. This projected motion is relevant in the study of light propagation in turbid media. Indeed, when persistent random walks are used as models for light propagation through slabs, the basic information is contained in the motion projected along the coordinate orthogonal to the faces of the slab [14]. We thus obtain the equation that governs the evolution of the projected motion and write its solution in terms of the free-space solution of the one-dimensional telegrapher's equation.

The paper is organized as follows. In Sec. II we set the general analysis for the PRW in a cubic lattice of dimension *d*. In Sec. III we obtain the continuum limit and find the governing equation for the probability density function of the resulting process. Section IV is devoted to the projected motion on a given coordinate and we obtain an exact expression of the density. Conclusions are drawn in Sec. V.

II. GENERAL ANALYSIS

We consider a random walk in a cubic lattice of arbitrary dimension *d*. The distance between nearest lattice points is *l*.

Jumps to another lattice point occur after a time interval τ. At each point, the random walker can take $2d$ different directions. Among all possible events, we are only interested in three of them: (a) The ''forward scattering,'' where the random walker moves in the same direction as the previous jump, (b) the ''backward scattering,'' where the walker reverses its previous direction, and (c) the scattering in the $2(d-1)$ remaining directions. Let us denote by α, β, and γ the probabilities of events (a), (b), and (c), respectively. As absorption will not be considered here, the scattering probabilities satisfy the normalization condition

$$\alpha+\beta+2(d-1)\gamma=1. \tag{2}$$

This kind of random walk is termed ''persistent'' because it introduces a persistent probability α and generalizes the one-dimensional PRW to arbitrary dimensions [3].

Let us now set the general equations of this model. We define a set of auxiliary probabilities $P_n^{(\pm k)}(i_1,\ldots,i_d)$, $k=1,\ldots,d$, where $P_n^{(+k)}(i_1,\ldots,i_d)$ is the probability that the walker reaches the lattice point $(i_1,\ldots,i_d)$ at step n moving along direction $+k$. A similar definition applies to $P_n^{(-k)}(i_1,\ldots,i_d)$. Following an analogous reasoning to that of the one-dimensional PRW [3], we can see that $P_n^{(\pm k)}(i_1,\ldots,i_d)$ obeys the following set of recursive equations:

$$P_{n+1}^{(+k)}(i_1,\ldots,i_d)=\alpha P_n^{(+k)}(i_1,\ldots,i_k-1,\ldots,i_d)+\beta P_n^{(-k)}(i_1,\ldots,i_k-1,\ldots,i_d)+\gamma\sum_{j\neq k}[P_n^{(+j)}(i_1,\ldots,i_k-1,\ldots,i_d)+P_n^{(-j)}(i_1,\ldots,i_k-1,\ldots,i_d)], \tag{3}$$

$$P_{n+1}^{(-k)}(i_1,\ldots,i_d)=\beta P_n^{(+k)}(i_1,\ldots,i_k+1,\ldots,i_d)+\alpha P_n^{(-k)}(i_1,\ldots,i_k+1,\ldots,i_d)+\gamma\sum_{j\neq k}[P_n^{(+j)}(i_1,\ldots,i_k+1,\ldots,i_d)+P_n^{(-j)}(i_1,\ldots,i_k+1,\ldots,i_d)]. \tag{4}$$

Let us briefly explain how these equations can be obtained. Consider $k=1$ and the origin located at $(0,\ldots,0)$. Suppose the random walker has reached the origin at step $n+1$ moving along direction $+1$. Then, it necessarily was at the lattice point $(-1,0,\ldots,0)$ at step n. The probability that the random walker jumps from this point to the origin depends on the arrival direction to the point $(-1,0,\ldots,0)$. Each direction (and there are $2d$ directions) contributes to the total probability $P_{n+1}^{(+1)}(0,\ldots,0)$ with a different weight. Indeed, the term $\alpha P_n^{(+1)}(-1,0,\ldots,0)$ gives the probability that the random walker arrives at $(-1,0,\ldots,0)$ along direction $+1$ and keeps going on the same direction. The probability that the random walker reverses its arrival direction after reaching $(-1,0,\ldots,0)$ is $\beta P_n^{(-1)}(-1,0,\ldots,0)$. The rest of the $2(d-1)$ directions contributes with the terms $\gamma P_n^{(\pm k)}(-1,0,\ldots,0)$, with $k\neq 1$. Equations (3) and (4) are easily obtained after generalizing this reasoning to arbitrary points $(i_1,\ldots,i_d)$ and directions $\pm k$. These equations completely characterize our extension of the PRW and they are a convenient starting point for numerical analysis when no further analytical treatment can be made.

III. THE CONTINUUM LIMIT

We now proceed to the continuum limit. In this situation the length of each step, l, and the time interval between jumps, τ, both go to zero in such a way that the random walker moves at finite velocity v,

$$v=\lim_{l,\tau\to 0}\frac{l}{\tau}. \tag{5}$$

As in the case of the one-dimensional PRW [3,4], we also have to scale the scattering probabilities in the form

$$\alpha=1-\lambda\tau,\quad \beta=c\tau,\quad \gamma=a\tau/2, \tag{6}$$

where τ is the interval between jumps and λ is a parameter whose units are $[T^{-1}]$. Let us now see what the physical meaning is of the parameters appearing in Eq. (6). We first observe that, as a direct consequence of the scaling (6), the occurrence of collision events is governed by the Poisson law. Indeed, in the discrete case the probability that the random walker jumps k times in the same direction is α^k. Therefore, in the continuum limit the probability that the particle keeps moving in the same direction for a time $t\geqslant k\tau$ is

$$\Psi(t)=\lim_{\tau\to 0}\alpha^k=\lim_{\tau\to 0}(1-\lambda\tau)^{t/\tau}=e^{-\lambda t}. \tag{7}$$

We thus see the physical meaning of the parameter λ defined in Eq. (6), since it represents the mean frequency at which the random walker changes its direction, that is, λ is the mean number of scattering events per unit time, or equivalently λ^{-1} is the mean time between collision events. As a consequence, since β is the probability of reversing direction, we see from Eq. (6) that c/λ is the conditional probability of reversing the direction of motion. Analogously $a/(2\lambda)$ is the probability that there is a turn to an orthogonal direction. Finally, the normalization condition requires that

$$c+(d-1)a=\lambda.$$

We also note that the angle between directions of motion before and after a collision is necessarily a multiple of $\pi/2$

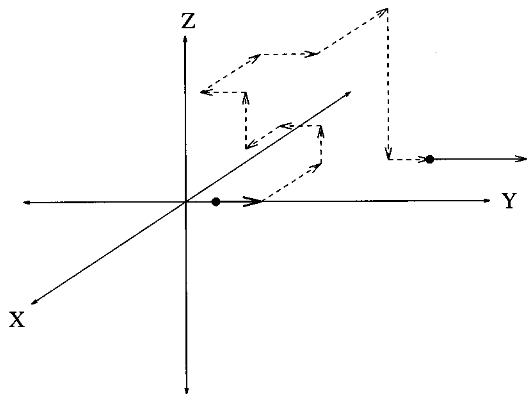

FIG. 1. A realization of the continuum model in three dimensions described in the paper. Note that the random walker moves along direcctions that are parallel to the axes.

(recall that the discrete model has been built on a cubic lattice). Therefore, the only possible velocities of the random walker are the $2d$ values $\pm v\hat{\boldsymbol{i}}_k$ $(k=1,\ldots,d)$, where $\hat{\boldsymbol{i}}_k$ is the unit vector in the x_k coordinate direction. Figure 1 shows a realization of the model in three dimensions.

In order to get the diffusive limit of Eqs. (3) and (4), we define $t=n\tau$ and $x_k=i_k l$, $k=1,\ldots,d$, and let $l\to 0$ and $\tau\to 0$ with the condition that v defined by Eq. (5) is constant. In this limit, probabilities $P_n^{(\pm)}(i_1,\ldots,i_d)$ become

$$P_{n+1}^{(\pm k)}(i_1,\ldots,i_j,\ldots,i_d) \to p^{(\pm k)}(\boldsymbol{r},t)+\tau\frac{\partial p^{(\pm k)}(\boldsymbol{r},t)}{\partial t}+O(\tau^2),$$

$$P_n^{(\pm k)}(i_1,\ldots,i_j\pm 1,\ldots,i_d) \to p^{(\pm k)}(\boldsymbol{r},t)\pm l\frac{\partial p^{(\pm k)}(\boldsymbol{r},t)}{\partial x_j}+O(l^2), \qquad (8)$$

where $p^{(\pm k)}(\boldsymbol{r},t)d\boldsymbol{r}$ is the joint probability density function for the position $\boldsymbol{r}=(x_1,\ldots,x_d)$ and the velocity of the random walker at time t. All the information about the continuous model is contained in the following set of equations:

$$\frac{\partial p^{(+k)}(\boldsymbol{r},t)}{\partial t}=-v\frac{\partial p^{(+k)}(\boldsymbol{r},t)}{\partial x_k}-\lambda p^{(+k)}(\boldsymbol{r},t)+cp^{(-k)}(\boldsymbol{r},t)+\frac{1}{2}a\sum_{j\neq k}[p^{(+j)}(\boldsymbol{r},t)+p^{(-j)}(\boldsymbol{r},t)],$$

$$\frac{\partial p^{(-k)}(\boldsymbol{r},t)}{\partial t}=v\frac{\partial p^{(-k)}(\boldsymbol{r},t)}{\partial x_k}-\lambda p^{(-k)}(\boldsymbol{r},t)+cp^{(+k)}(\boldsymbol{r},t)+\frac{1}{2}a\sum_{j\neq k}[p^{(+j)}(\boldsymbol{r},t)+p^{(-j)}(\boldsymbol{r},t)], \qquad (9)$$

which are the result of applying the continuum limit (8) to the general recursive equations (3) and (4).

In many applications the most interesting quantity is the probability density $p(\boldsymbol{r},t)$ for the position independent of the velocity, that is,

$$p(\boldsymbol{r},t)=\sum_{k=1}^{d}[p^{(+k)}(\boldsymbol{r},t)+p^{(-k)}(\boldsymbol{r},t)]. \qquad (10)$$

For $d=1$, we already explained that $p(x,t)$ evolves according to a TE, Eq. (1). We now present the partial differential equation describing the evolution of this density when $d\geqslant 2$. We show in Appendix A that this equation reads

$$[\partial_t(\partial_t+b)^d(\partial_t+ad)^{d-1}-(\partial_t+b)^{d-1}\times(\partial_t+ad)^{d-2}(\partial_t+a)v^2\nabla^2-\Phi_d]p(\boldsymbol{r},t)=0, \qquad (11)$$

where ∇^2 is the Laplacian of the spatial coordinates, $b=\lambda+c$, and Φ_d is an operator including all spatial partial derivatives of fourth order or greater. For $d=2$ and $d=3$, the operator Φ_d has the following expressions:

$$\Phi_2=-v^4\partial^4_{x^2y^2}, \qquad (12)$$

$$\Phi_3=v^6\partial^6_{x^2y^2z^2}-v^4(\partial_t+b)(\partial_t+2a)(\partial^4_{x^2y^2}+\partial^4_{x^2z^2}+\partial^4_{y^2z^2}). \qquad (13)$$

We have thus obtained the $2d$ order partial differential equation that satisfies the probability density function of the persistent random walk in higher dimensions. This equation does not show spherical symmetry because of the velocities allowed by the model. Nevertheless, using the so-called ''dominant balance technique'' [18] one can easily see that the behavior of the probability density function at long times, and for positions $\boldsymbol{r}$ sufficiently far away from the moving boundary, is given by the lowest-order partial derivatives. This ''central limit approximation'' transforms Eq. (11) into the diffusion equation

$$\partial_t p(\boldsymbol{r},t)=D\nabla^2 p(\boldsymbol{r},t), \qquad (14)$$

where the diffusion constant D is

$$D=\frac{v^2}{bd}. \qquad (15)$$

Note that the diffusion constant is in agreement with the result that follows from transport theory $D=vl_t/d$, where $l_t=v/b$ is the transport mean free path [15].

IV. THE PROJECTED MOTION

Let us now study the projected motion of the PRW on a given axis. The probability density function of this motion is given by

$$p(x,t)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}p(\boldsymbol{r},t)dx_2\cdots dx_d.$$

This marginal density obeys a simpler equation than Eq. (11) mainly because the integration of $\Phi_d p(\boldsymbol{r},t)$ over the coordinates $x_2\cdots x_d$ is zero. We show in Appendix B that the equation for $p(x,t)$ reads

$$\partial_t(\partial_t+b)(\partial_t+ad)p(x,t)=v^2(\partial_t+a)\partial^2_{x^2}p(x,t). \qquad (16)$$

This equation is a third-order hyperbolic partial differential equation with finite velocity of propagation v. The reduction in the order from $2d$ for Eq. (11) to 3 for Eq. (16) is due to the fact that the projected motion is coupled to the motion along any other direction by a first-order equation (see Appendix B). Equation (16) may provide a better approximation to transport problems than the diffusion equation, especially in thin slabs where the effect of taking into account the speed of propagation is crucial [16]. Note that Eq. (16) possesses a richer structure than the telegrapher's equation. Moreover, we have been able to derive Eq. (16) from a microscopic model (we recall that any attempt to derive TE from a microscopic model of transport fails in higher dimensions). Nevertheless, Eq. (16) shares with TE at least two important features: (i) a finite velocity of propagation, and (ii) a similar asymptotic behavior of the moments (see below). An equation similar to Eq. (16) was used several years ago in the context of heat propagation in rigid solids [17].

It is possible to exactly solve Eq. (16) in the Fourier-Laplace space for the isotropic initial conditions

$$p^{(+k)}(\boldsymbol{r},t)=p^{(-k)}(\boldsymbol{r},t)=\frac{1}{2d}\delta(\boldsymbol{r}), \tag{17}$$

which provide the three initial conditions required for solving Eq. (16),

$$\begin{aligned} p(x,t=0)&=\delta(x),\\ \left.\frac{\partial p(x,t)}{\partial t}\right|_{t=0}&=0,\\ \left.\frac{\partial^2 p(x,t)}{\partial t^2}\right|_{t=0}&=\frac{v^2}{d}\delta''(x). \end{aligned} \tag{18}$$

In Appendix C we show how to derive these conditions from Eq. (17). It is possible to solve Eq. (16) in the Fourier-Laplace space. To this end we define the joint transform

$$\hat{p}(\omega,s)\equiv\int_{-\infty}^{\infty}\int_0^{\infty}e^{-st}e^{-i\omega x}p(x,t)dx\,dt.$$

Then the transformation of Eq. (16) and the use of Eq. (18) lead to the solution

$$\hat{p}(\omega,s)=\frac{(s+ad)(s+b)+v^2\omega^2(1-1/d)}{s(s+ad)(s+b)+v^2\omega^2(s+a)}. \tag{19}$$

We observe that this case is equivalent to a three-state continuous-time random walk where the particle is moving to the right (or left) with velocity v $(-v)$ or is at rest, which corresponds to the motion in an orthogonal direction. The transition matrix of this three-state walk is [3]

$$p(i\to j)=\begin{pmatrix} 0 & c/\lambda & (d-1)a/\lambda\\ c/\lambda & 0 & (d-1)a/\lambda\\ a/(2\lambda) & a/(2\lambda) & 1-a/\lambda \end{pmatrix}, \tag{20}$$

where $i=1$ corresponds to the state with velocity $+v$, $i=2$ with velocity $-v$, and $i=3$ to the state without displacement along the x direction.

Let us now study the moments $m_n(t)$, $n=1,2,3,\ldots$, of the distribution $p(x,t)$. In terms of the characteristic function, $\tilde{p}(\omega,t)$, the moments are given by $m_n(t)=i^n\partial_\omega\hat{p}(\omega,t)|_{\omega=0}$. Due to the isotropic initial conditions, Eq. (18), all odd moments vanish, $m_{2n-1}(t)=0$. Then it follows from Eq. (19) that the Laplace transform of $m_{2n}(t)$ is given by

$$\hat{m}_{2n}(s)=\frac{(2n)!}{d\,s}\left[\frac{s+a}{(s+da)}\right]^{n-1}\left[\frac{v^2}{s(s+b)}\right]^n. \tag{21}$$

The leading behavior for small s is

$$\hat{m}_{2n}(s)\sim\frac{v^{2n}(2n)!}{b^nd^n}s^{-1-n}\quad(s\to0)$$

and the leading behavior for large s is

$$\hat{m}_{2n}(s)\sim\frac{v^{2n}(2n)!}{d}s^{-1-2n}\quad(s\to\infty).$$

Then, by the Tauberian theorems [3,18], we get the asymptotic behaviors

$$m_{2n}(t)\sim\frac{(2n)!}{n!}\left(\frac{v^2}{bd}t\right)^n\quad(t\to\infty) \tag{22}$$

and

$$m_{2n}(t)\sim\frac{v^{2n}}{d}t^{2n}\quad(t\to0). \tag{23}$$

Note that the moments as $t\to\infty$ are identical to those of ordinary diffusion with $D=v^2/bd$ in agreement with the result of the preceding section, which means that $p(x,t)$ behaves in this limit as

$$p(x,t)\sim\frac{1}{\sqrt{4\pi Dt}}\exp[-x^2/4Dt]\quad(t\to\infty). \tag{24}$$

On the other hand, when $t\to0$, the motion behaves as if it were deterministic. In fact, taking into account the isotropic initial conditions, we can easily show that the density $p(x,t)$ has the following expansion at short times:

$$p(x,t)\sim\frac{1}{2d}\delta(x-vt)+\frac{1}{2d}\delta(x+vt)+\frac{d-1}{d}\delta(x)\quad(t\to0), \tag{25}$$

which immediately leads to the result (23) for the moments. Therefore, the behavior of moments is similar to that of the solution of the telegrapher's equation. Indeed, we see from Eqs. (22) and (23) that if $t\to\infty$, we have ordinary diffusion (where, for instance, the second moment goes as t), while if $t\to0$ the behavior is deterministic (where the second moment goes as t^2).

We have not been able to invert Eq. (19) in general. However, the solution in real space can be written for the two-dimensional case $d=2$ and when the backscattering probability vanishes, $c=0$. In effect, now $a=b=\lambda$, and the expression for $\hat{p}(\omega,s)$ reads

$$\hat{p}(\omega,s)=\frac{1}{2(s+\lambda)}+\frac{1}{2}\ \hat{p}_{te}(\omega,s)+\frac{1}{2(s+\lambda)}\ \hat{p}_{te}(\omega,s), \tag{26}$$

where

$$\hat{p}_{te}(\omega,s)\equiv\frac{s+2\lambda}{s(s+2\lambda)+v^2\omega^2} \tag{27}$$

is the Fourier-Laplace transform of the free-space solution of the one-dimensional telegrapher's equation for isotropic initial conditions. Indeed, the Fourier-Laplace transform of Eq. (1) along with the isotropic initial conditions $p_{te}(x,0)=\delta(x)$ and $\partial_t p_{te}(x,0)=0$ leads to Eq. (27) [6,19]. Therefore, we see from Eq. (26) that the sought-after expression reads

$$p(x,t)=\frac{1}{2}e^{-\lambda t}\delta(x)+\frac{1}{2}p_{te}(x,t)+\frac{\lambda}{2}e^{-\lambda t}\int_0^t e^{\lambda\tau}p_{te}(x,\tau)d\tau, \tag{28}$$

where $p_{te}(x,t)$ is [4,19]

$$\begin{aligned}p_{te}(x,t)=&\frac{e^{-\lambda t}}{2v}\delta(t-|x|/v)+\frac{\lambda e^{-\lambda t}}{2v}\Theta(t-|x|/v)\\&\times\Bigg\{I_0(\lambda\sqrt{t^2-x^2/v^2})\\&+\frac{t}{\sqrt{t^2-x^2/v^2}}I_1(\lambda\sqrt{t^2-x^2/v^2})\Bigg\},\end{aligned} \tag{29}$$

where $I_n(z)$ are modified Bessel functions. Surprisingly, $p_{te}(x,\tau)$ is also the solution of a three-state random walk with transition matrix (20) when the initial conditions have an equal probability of moving to the left or to the right, and zero probability to be at rest. This fact clarifies the origin of solution (28). Indeed, the initial conditions for that solution were 1/2 probability to be at rest and 1/4 to move in either x direction. The δ term in $p(x,t)$ accounts for the probability that the random walker keeps moving along the y axis for all period t. The second term is the contribution of random walkers that are not at rest initially, according to the interpretation for $p_{te}(x,\tau)$ given above. Finally, particles at rest at $t=0$ that begin to move at some time between $(0,t)$ can be considered as a source of particles that evolve as $p_{te}(x,\tau)$. The third term is therefore the convolution of the source function $(\lambda e^{-\lambda t})$ with $p_{te}(x,\tau)$.

In Fig. 2, we plot the solution for different times (without δ terms). When t is less than λ^{-1}, the δ-function terms account for most of $p(x,t)$. As time grows, the contribution of these terms decays exponentially and the solution converges to the Gaussian distribution, Eq. (24).

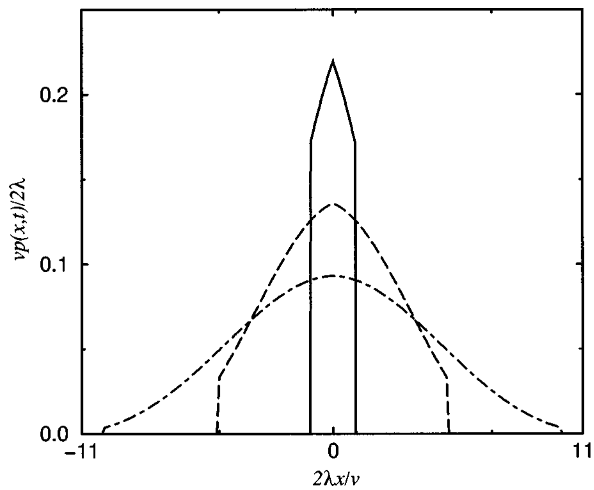

FIG. 2. Probability density function $p(x,t)$ for the motion projected along the x direction at different times as a function of the position x. Solid line: $2xt=1$; dashed line: $2\gamma t=5$; dot-dashed line: $2\gamma t=10$.

V. CONCLUSIONS

The generalization of the persistent random walk to dimensions higher than 1 does not evolve according to a telegrapher's equation as it does in one dimension. We have explicitly obtained the $2d$ order partial differential equation governing the evolution of the probability density function. Therefore, higher-dimensional persistent random walks in cubic lattices cannot be considered as microscopic models to derive TE in dimensions greater than 1.

Moreover, even the motion projected along an axis, besides being one-dimensional, does not evolve according to a TE. We have found that the probability density of the projection obeys a third-order partial differential equation. The order of the equation is completely consistent with the fact that the projected motion is equivalent to a three-state random walk.

The evolution equation for the projected motion may be very useful when considered as an approximation to the full transport equation for models of light propagation because Eq. (16) may overcome the problems of the diffusive approximation when light propagates through thin slabs. In this sense we have shown that the evolution equation for the projected motion correctly links the expected short-time behavior (deterministic) with the long-time behavior (diffusive) of these models.

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APPENDIX A: DERIVATION OF EQ. (11)

Let us define the following functions:

$$\begin{aligned}U_k(\boldsymbol{r},t)&\equiv p^{(+k)}(\boldsymbol{r},t)+p^{(-k)}(\boldsymbol{r},t),\\V_k(\boldsymbol{r},t)&\equiv p^{(+k)}(\boldsymbol{r},t)-p^{(-k)}(\boldsymbol{r},t),\end{aligned} \tag{A1}$$

for $k=1,\dots,d$. The probability density of the position becomes

$$p(\boldsymbol{r},t)=\sum_{k=1}^{d} U_k(\boldsymbol{r},t), \tag{A2}$$

and the set of Eqs. (9) reads

$$\frac{\partial U_k(\boldsymbol{r},t)}{\partial t}=-v\frac{\partial V_k(\boldsymbol{r},t)}{\partial x_k}-adU_k(\boldsymbol{r},t)+ap(\boldsymbol{r},t),$$

$$\frac{\partial V_k(\boldsymbol{r},t)}{\partial t}=-v\frac{\partial U_k(\boldsymbol{r},t)}{\partial x_k}-bV_k(\boldsymbol{r},t), \tag{A3}$$

where $b=\lambda+c$. If we define the operator

$$\mathcal{D}_\rho\equiv\partial_t+\rho,$$

where ρ is a constant, then the equation for $U_k(\boldsymbol{r},t)$ becomes

$$\mathcal{D}_b[ap(\boldsymbol{r},t)-\mathcal{D}_{ad}U_k(\boldsymbol{r},t)]=-v^2\frac{\partial^2 U_k(\boldsymbol{r},t)}{\partial x_k^2},\quad k=1,\dots,d. \tag{A4}$$

For $d=1$, this equation coincides with the telegrapher's equation because, in this case, $a=0$ and $U_1(x,t)=p(x,t)$. For $d\geqslant 2$, the definition of the operator

$$\mathcal{S}_k\equiv\mathcal{D}_b\mathcal{D}_{ad}-v^2\partial_{x_k}^2$$

further simplifies Eq. (A4) into

$$\mathcal{S}_kU_k(\boldsymbol{r},t)=a\mathcal{D}_bp(\boldsymbol{r},t). \tag{A5}$$

Finally, applying $\Pi_{i=1}^{d}\mathcal{S}_i$ to both sides of Eq. (A2) and using Eq. (A5), we obtain the equation for $p(\boldsymbol{r},t)$,

$$\prod_{i=1}^{d}\mathcal{S}_ip(\boldsymbol{r},t)=a\mathcal{D}_b\sum_{k=1}^{d}\prod_{i\neq k}\mathcal{S}_ip(\boldsymbol{r},t). \tag{A6}$$

The expansion of the product $\Pi_{i=1}^{d}\mathcal{S}_i$ in powers of $\mathcal{D}_b\mathcal{D}_{ad}$ leads to Eq. (11).

APPENDIX B: DERIVATION OF EQ. (16)

The integration of the system (A3) over the coordinates $x_2\cdots x_d$ yields

$$\frac{\partial U_k(x,t)}{\partial t}=-adU_k(x,t)+ap(x,t), \tag{B1}$$

$$\frac{\partial V_k(x,t)}{\partial t}=-bV_k(x,t),$$

for $k=2,\dots,d$ and

$$\frac{\partial U_1(x,t)}{\partial t}=-v\frac{\partial V_1(x,t)}{\partial x}-adU_1(x,t)+ap(x,t),$$

$$\frac{\partial V_1(x,t)}{\partial t}=-v\frac{\partial U_1(x,t)}{\partial x}-bV_1(x,t), \tag{B2}$$

where x_1 was denoted by x and $U_k(x,t)$ and $V_k(x,t)$ correspond to the functions $U_k(\boldsymbol{r},t)$ and $V_k(\boldsymbol{r},t)$ integrated over the coordinates $x_2\cdots x_d$. The first equation of system (B1) implies that the quantity

$$q(x,t)\equiv\sum_{k=2}^{d} U_k(x,t)$$

satisfies the first-order equation

$$\frac{\partial q(x,t)}{\partial t}=-adq(x,t)+a(d-1)[U_1(x,t)+q(x,t)]. \tag{B3}$$

This equation along with system (B2) leads to a third-order partial differential equation for

$$p(x,t)=U_1(x,t)+q(x,t).$$

Indeed, from Eqs. (B1) and (B2) we get

$$\partial_tD_bp(x,t)=v^2\partial_x^2U_1(x,t) \tag{B4}$$

and

$$D_{ad}U_1(x,t)=D_ap(x,t). \tag{B5}$$

Finally, the combination of Eqs. (B4) and (B5) immediately leads to Eq. (16).

APPENDIX C: INITIAL CONDITIONS

The integration of the isotropic initial conditions (17) over the coordinates $x_2\cdots x_d$ gives the initial conditions for the marginal probability

$$p^{(+k)}(x,t)=p^{(-k)}(x,t)=\frac{1}{2d}\delta(x).$$

The first initial condition follows directly from them,

$$p(x,t=0)=\sum_{k=1}^{d}[p^{(+k)}(x,t=0)+p^{(-k)}(x,t=0)]=\delta(x).$$

Taking into account that $V_k(x,t=0)=0$, we see from Eq. (A3) that

$$\left.\frac{\partial U_k(x,t)}{\partial t}\right|_{t=0}=-ad\frac{1}{d}\delta(x)+a\delta(x)=0, \tag{C1}$$

and therefore the second initial condition reads

$$\left.\frac{\partial p(x,t)}{\partial t}\right|_{t=0}=0. \tag{C2}$$

After taking the derivative with respect to time in the equation for $U_k(\boldsymbol{r},t)$ of system (A3) and using the equation for $V_k(\boldsymbol{r},t)$ in the same system, we get

$$\frac{\partial^2 U_k(\boldsymbol{r},t)}{\partial t^2} = v^2 \frac{\partial^2 U_k(\boldsymbol{r},t)}{\partial x_k^2} + bv \frac{\partial V_k(\boldsymbol{r},t)}{\partial x_k} - adU_k(\boldsymbol{r},t) + ap(\boldsymbol{r},t).$$

The sum over all k and the integration over coordinates $x_2 \cdots x_d$ of this equation gives the third initial condition

$$\left.\frac{\partial^2 p(x,t)}{\partial t^2}\right|_{t=0} = \frac{v^2}{d} \delta''(x).$$

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