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Pacemakers in a Cayley tree of Kuramoto oscillators

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Abstract. In this work we study a system of Kuramoto oscillators with identical frequencies in a Cayley tree. Heterogeneity in the frequency distribution is introduced in the root of the tree, allowing for analytical calculations of the phases evolution.

Keywords: Synchronization, Cayley Tree, Kuramoto model
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1. Introduction

The heterogeneous structure of complex networks plays a fundamental role in the synchronization properties in systems of coupled Kuramoto oscillators Kori and Mikhailov (2004); Radicchi and Meyer-Ortmanns (2006); Arenas et al. (2008); Buzna et al. (2009). In order to understand the interplay between network structure and synchronization dynamics we focus our interest in a simple topology, the Cayley tree. Taking as a reference a system of identical oscillators we introduce heterogeneity by changing the natural frequency of the oscillator located at the root of the Cayley tree. In this way we are able to obtain analytic results for the phase evolution of the oscillators.

2. The model

We consider as the network structure of our system a Cayley tree of variable radii and coordination numbers. The nodes follow the dynamics described by the Kuramoto model of phase oscillators Kuramoto (1984); Acebrón et al. (2005). All of them have the same natural frequency (taken to be zero without loss of generality) except for the one located at the root of the tree, that has a variable frequency. We will refer to this node as the pacemaker. The equations are

$$\dot{\phi}_p = \omega + \sum_j a_{pj} \sin(\phi_j - \phi_p); \quad \dot{\phi}_i = \sum_j a_{ij} \sin(\phi_j - \phi_i) \tag{1}$$

where the first one is for the pacemaker, the second for all the rest. The matrix a_{ij} is the corresponding adjacency matrix for the Cayley tree.

We analyze an order parameter that measures the effective frequency dispersion

$$\Delta_\omega = \sqrt{\frac{1}{N}\sum_{i\in N}[\dot{\phi_i} - <\omega>]^2} \tag{2}$$

In Fig. 1 we present the time evolution of this order parameter for a Cayley tree with a coordination number $q = 3$, and radius $R = 5$, with a total of $N = 94$ nodes. The results correspond to initial conditions with all the phases equal to zero. For frequencies below a critical value ω_c the order parameter decays exponentially to zero, revealing that for long times there is no dispersion in the effective frequencies and the system becomes synchronized. Above ω_c the system does not reach equilibrium, and the order parameter presents an oscillating behavior.

Note also that the frequency dispersion presented in Eq. (2) is not normalized. We divide the frequency dispersion by its maximum allowed value $\frac{\omega}{N}\sqrt{N-1}$, and thus obtain a normalized *order parameter*:

$$r_\omega = \sqrt{\frac{1}{N-1}\sum_{i\in N}\left[\frac{\dot{\phi_i}}{\langle\omega\rangle} - 1\right]^2} \tag{3}$$

In Fig. 1 (right) we present the stationary value of the normalized order parameter as a function of the pacemaker frequency ω. As noted, for values above ω_c the system is not in equilibrium, however it is possible to define a mean value around which the order parameter oscillates. The figure shows the behavior observed for three different values of the coordination number, $q = 3$, 4 and 5. As q grows, the critical value also grows accordingly. The inset in the figure shows the data collapse obtained by scaling the natural frequency of the pacemaker with the coordination number, ω/q.

3. Results

The order parameter defined by Kuramoto is $z(t) = \frac{1}{N}\sum_j \exp(i\phi_j)$. Following the calculation of Ref. Prignano and Díaz-Guilera (2010) an upper bound for the critical condition for synchronization of the pacemaker with the system is $\phi_j - \phi_p = -\frac{\pi}{2}$, where j are the first neighbors of the pacemaker. If we consider a star shaped network with the pacemaker in the center, then this last equation becomes

$$z(t) = \frac{1}{N}[\exp(i\phi_p)(1 - ik_p)] \tag{4}$$

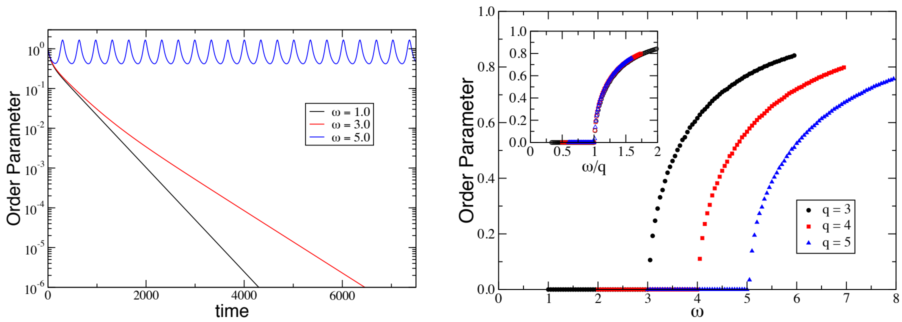

Figure 1: (Left) Effective Frequency Dispersion Order Parameter vs. time. (Right) Normalized Frequency Dispersion Order Parameter vs. natural frequency of the pacemaker ω.

and we have an analytical expression for the behavior of the Kuramoto order parameter. While for the effective frequencies we obtain:

$$\dot{\phi}_{i\neq p} = -\sin\left(2\arctan\left[A\tan\left(-\frac{A}{2}t + B\right) - N\right]\right); \tag{5}$$

and

$$\dot{\phi}_p = \omega + (1-N)\dot{\phi}_{i\neq p}, \tag{6}$$

where $A = \sqrt{\omega^2 - N}$ and $B = \arctan(\frac{N}{A})$. Note however, that if we consider a Cayley tree with increasing radius then the last term becomes clearly dominant, and eventually, the perturbation introduced by the pacemaker will not be detected by the order parameter.

4. Conclusions

Taking a Cayley tree as a reference structure, we have computed analytically the effect of a single pacemaker located at the root of the tree for a system of Kuramoto oscillators.

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